Exercise 3.1 - Treaps

1. Insert $f$: \[
\begin{array}{c}
\frac{7}{7}
\end{array}
\]

Insert $g$: \[
\begin{array}{c}
\frac{6}{6} \\
\frac{7}{7}
\end{array}
\rightarrow
\begin{array}{c}
\frac{6}{6} \\
\frac{7}{7}
\end{array}
\]

Insert $h$: \[
\begin{array}{c}
\frac{25}{25} \\
\frac{7}{7}
\end{array}
\]

Insert $e$: \[
\begin{array}{c}
\frac{3}{3} \\
\frac{25}{25} \\
\frac{7}{7} \\
\frac{6}{6}
\end{array}
\rightarrow
\begin{array}{c}
\frac{3}{3} \\
\frac{25}{25} \\
\frac{7}{7} \\
\frac{6}{6}
\end{array}
\rightarrow
\begin{array}{c}
\frac{3}{3} \\
\frac{25}{25} \\
\frac{7}{7} \\
\frac{6}{6}
\end{array}
\rightarrow
\begin{array}{c}
\frac{3}{3}
\end{array}
\]

Insert $b$: \[
\begin{array}{c}
\frac{15}{15} \\
\frac{6}{6}
\end{array}
\rightarrow
\begin{array}{c}
\frac{15}{15} \\
\frac{6}{6}
\end{array}
\rightarrow
\begin{array}{c}
\frac{15}{15} \\
\frac{6}{6}
\end{array}
\rightarrow
\begin{array}{c}
\frac{15}{15}
\end{array}
\]
Insert $a$:

$\begin{array}{c}
\begin{array}{cccc}
e & e & e \\
3 & 3 & 3 \\
\end{array} \\
\begin{array}{cccc}
\downarrow & \downarrow & \downarrow \\
b & g & \rightarrow \\
15 & 6 & 25 \\
\end{array} \\
\begin{array}{cccc}
a & f & h \\
8 & 7 & 25 \\
\end{array} \\
\begin{array}{cccc}
\downarrow & \downarrow & \downarrow \\
c & b & g \\
2 & 15 & 6 \\
\end{array}
\end{array}$

Insert $c$:

$\begin{array}{c}
\begin{array}{cccc}
e & e & e \\
3 & 3 & 3 \\
\end{array} \\
\begin{array}{cccc}
\downarrow & \downarrow & \downarrow \\
c & a & g \\
2 & 8 & 6 \\
\end{array} \\
\begin{array}{cccc}
b & f & h \\
15 & 7 & 25 \\
\end{array} \\
\begin{array}{cccc}
\downarrow & \downarrow & \downarrow \\
c & b & g \\
2 & 15 & 6 \\
\end{array}
\end{array}$
2. Delete c:  

\[
\begin{array}{c}
\text{a} \\
\frac{8}{2} \\
\text{b} \\
\frac{15}{3} \\
\text{c} \\
\frac{6}{3} \\
\text{d} \\
\frac{7}{25} \\
\text{e} \\
\frac{25}{25} \\
\end{array}
\]

\[
\begin{array}{c}
\text{a} \\
\frac{8}{2} \\
\text{b} \\
\frac{15}{3} \\
\text{c} \\
\frac{6}{3} \\
\text{d} \\
\frac{7}{25} \\
\text{e} \\
\frac{25}{25} \\
\end{array}
\]

\[
\begin{array}{c}
\text{a} \\
\frac{8}{2} \\
\text{b} \\
\frac{15}{3} \\
\text{c} \\
\frac{6}{3} \\
\text{d} \\
\frac{7}{25} \\
\text{e} \\
\frac{25}{25} \\
\end{array}
\]

\[
\begin{array}{c}
\text{a} \\
\frac{8}{2} \\
\text{b} \\
\frac{15}{3} \\
\text{c} \\
\frac{6}{3} \\
\text{d} \\
\frac{7}{25} \\
\text{e} \\
\frac{25}{25} \\
\end{array}
\]

\[
\begin{array}{c}
\text{a} \\
\frac{8}{2} \\
\text{b} \\
\frac{15}{3} \\
\text{c} \\
\frac{6}{3} \\
\text{d} \\
\frac{7}{25} \\
\text{e} \\
\frac{25}{25} \\
\end{array}
\]

\[
\begin{array}{c}
\text{a} \\
\frac{8}{2} \\
\text{b} \\
\frac{15}{3} \\
\text{c} \\
\frac{6}{3} \\
\text{d} \\
\frac{7}{25} \\
\text{e} \\
\frac{25}{25} \\
\end{array}
\]

\[
\begin{array}{c}
\text{a} \\
\frac{8}{2} \\
\text{b} \\
\frac{15}{3} \\
\text{c} \\
\frac{6}{3} \\
\text{d} \\
\frac{7}{25} \\
\text{e} \\
\frac{25}{25} \\
\end{array}
\]

3. The result for insert i, k, j is:  

\[
\begin{array}{c}
k \\
\frac{13}{13} \\
j \\
\frac{19}{19} \\
i \\
\frac{19}{19} \\
\end{array}
\]

We choose ha with \(\text{prio}(ha) = -\infty\) as root, and the two heaps (1) and (2) as children. Afterwards we remove ha from the treap.
4. We insert $d$ with $\text{prio}(d) = -\infty$ into (3).
Hence we get the two treaps:

Exercise 3.2 - Treaps

1. Proving that $C + D$ is equal to the number of rotations, we have a look at the treap, when an arbitrary number of rotations is already done. We assume that $C + D$ holds, and we prove that $C + D$ holds for the treap after a rotation, (if necessary).

Consider the following treap:
The picture shows that the length of the new right spline of the subtree of $x$ is $C' = C + 1$, since it contains all elements the right spine of $t_2$ and $u$, while the length of the left spine $t_3$ remains unchanged.

Hence, after the rotation $C' + D' = C + D + 1$.

The other rotation is symmetric, there it holds $C' = C$ and $D' = D + 1$, hence $C' + D' = C + D + 1$.

Because $C + D$ is initially 0, after all rotations are done, $C + D$ is equal to the number of rotations that was needed to move $x$ to the correct position in the treap.

2. We insert $x$ with priority $\infty$ into $T_S$. The new treap $T_{S'}$ is the treap with elements $S' = S \cup \{x_1, \ldots, x_m, x, x_{m+1}, \ldots, x_n\} = \{x'_1, \ldots, x'_{n+1}\}$.

The expected number of nodes on the search path of $x$ in $T_{S'}$ is equal to the number of visited nodes for searching $x$ in $T_S$.

Define
\[
X_{k,i} = \begin{cases} 
1 & x'_i \text{ is ancestor of } x'_k \\
0 & \text{otherwise} 
\end{cases}
\]

Let $X_k$ be the length of the path from the root to $x'_k$.

\[
X_{m+1} = \sum_{i \in \{1, \ldots, n+1\}} X_{m+1,i} = \sum_{i < m+1} X_{m+1,i} + \sum_{i > m+1} X_{m+1,i}
\]

\[
\Rightarrow E[X_{m+1}] = \sum_{i < m+1} E[X_{m+1,i}] + \sum_{i > m+1} E[X_{m+1,i}]
\]

Since for all $x$: $\text{prio}(x_{m+1}) > \text{prio}(x)$

\[
i < m + 1 : \quad E[X_{m+1,i}] = P[x'_i \text{ is ancestor of } x'_{m+1}] = P[\text{prio}(\{x'_1, \ldots, x'_{m+1}\}) = x'_i] = P[\text{prio}(\{x'_i, \ldots, x'_m\}) = x'_i] = \frac{1}{m - i + 1}
\]

\[
i > m + 1 : \quad E[X_{m+1,i}] = P[x'_i \text{ is ancestor of } x'_{m+1}] = P[\text{prio}(\{x'_{m+1}, \ldots, x'_i\}) = x'_i] = P[\text{prio}(\{x'_m+2, \ldots, x'_i\}) = x'_i] = \frac{1}{i - (m + 2) + 1}
\]

\[
E[X_{m+1}] = \sum_{i < m+1} \frac{1}{m - i + 1} + \sum_{i > m+1} \frac{1}{i - (m + 2) + 1} = \frac{m}{1} + \sum_{i=1}^{n-m} \frac{1}{i} = H_m + H_{n-m}
\]
**Exercise 3.3 - Universal hashing**

\[ H = \{ h_i(k) = ((a_i \cdot k + b_i) \mod N - 1) \mod m \} \text{ for } i \in \{1, \ldots, N(N-1)\} \]

Since \( a_i = 40 \cdot i \) and \( b_i = 60 \cdot i \) are always even numbers, we can represent each \( h_i \in H \) as

\[ h_i(k) = (2 \cdot (a_i \cdot k^2 + b_i^2) \mod N - 1) \mod m. \]

In addition, since \( N \) is a prime number, \( N - 1 \) is an even number. Thus, every \( h_i \in H \) yields an even number. Therefore, the total number of cells used by \( H \) is \( \frac{m^2}{2} \).

Now, let \( \delta_h(x, y) \) and \( x, y \in U \) be defined as in the lecture (slide 17.) as follows:

\[ \delta_h(x, y) = \begin{cases} 1 & h(x) = h(y) \text{ and } x \neq y \\ 0 & \text{otherwise} \end{cases} \]

\( H \) is universal, if the following condition holds:

\[ \sum_{h \in H} \delta_h(x, y) \leq \frac{N(N-1)}{m} \]

Which implies that \( H \) is not universal.

In order to ensure \( H \) to be universal, we modify each \( h_i \in H \) with the help of the theorem from slide 18, as follows. \( a_i = (i - 1) \mod (N - 1) + 1 \), \( b_i = \left\lfloor \frac{N - 1}{N - 1} \right\rfloor \) and \( h_i(k) = ((a_i \cdot k + b_i) \mod N) \mod m. \) Now, \( H \) is a Universal class of hash functions, since it holds:

\[ \{(a_i, b_i) \mid i \in \{1, \ldots, N(N-1)\} \} = \{(a, b) \mid a \in \{1, \ldots, N - 1\}, b \in \{0, \ldots, N - 1\}\} \]

**Exercise 3.4 - Perfect hashing**

\( U = \{0, \ldots, 30\}, S = \{2, 4, 7, 11, 12, 18, 19, 21, 26, 28\} \) and \( k = 3, N = 31, n = |S| = 10. \)

1. First hash function \( h_3(x) = ((3x) \mod 31) \mod 10 \)

<table>
<thead>
<tr>
<th>i</th>
<th>( W_i = { x \in S : h_3(x) = i } )</th>
<th>( b_i )</th>
<th>( m_i = 2b_i(l_i - 1) + 1 )</th>
<th>( s_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \emptyset )</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>7,21</td>
<td>2</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4,11,28</td>
<td>3</td>
<td>13</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
<td>1</td>
<td>1</td>
<td>19</td>
</tr>
<tr>
<td>4</td>
<td>( \emptyset )</td>
<td>0</td>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td>1</td>
<td>1</td>
<td>21</td>
</tr>
<tr>
<td>6</td>
<td>2,19,26</td>
<td>3</td>
<td>13</td>
<td>22</td>
</tr>
<tr>
<td>7</td>
<td>( \emptyset )</td>
<td>0</td>
<td>1</td>
<td>35</td>
</tr>
<tr>
<td>8</td>
<td>( \emptyset )</td>
<td>0</td>
<td>1</td>
<td>36</td>
</tr>
<tr>
<td>9</td>
<td>( \emptyset )</td>
<td>0</td>
<td>1</td>
<td>37</td>
</tr>
</tbody>
</table>
Now compute $k_i$, such that $h_{k_i}(x) = ((k_i x) \mod 31) \mod m_i$ restricted to $W_i$ is injective.

- $W_1$

  \[
  \begin{array}{|c|c|}
  \hline
  x & h_1(x) \\
  \hline
  7 & 2 \\
  21 & 1 \\
  \hline
  \end{array}
  \]

  For $k_1 = 1$, $h_{k_1}$ restricted to $W_1$ is injective.

- $W_2$

  \[
  \begin{array}{|c|c|}
  \hline
  x & h_{k_2}(x) \\
  \hline
  4 & 4 \\
  11 & 11 \\
  28 & 2 \\
  \hline
  \end{array}
  \]

  For $k_2 = 1$, $h_{k_2}$ restricted to $W_2$ is injective.

- $W_6$

  \[
  \begin{array}{|c|c|}
  \hline
  x & h_{k_6}(x) \\
  \hline
  2 & 2 \\
  19 & 6 \\
  26 & 0 \\
  \hline
  \end{array}
  \]

  For $k_6 = 1$, $h_{k_6}$ restricted to $W_6$ is injective.

Then we obtain the following hash functions for the second level.

\[
\begin{align*}
  h_{k_1} &= (x \mod 31) \mod 5 \\
  h_{k_2} &= (x \mod 31) \mod 13 \\
  h_{k_6} &= (x \mod 31) \mod 13 \\
  h_{k_j} &= (x \mod 31) \mod 1 \quad \text{for } j \in \{0, \ldots, n - 1\} \setminus \{1, 2, 6\}
\end{align*}
\]

2. Positions of the elements $x \in S$ in the hash table: $\text{pos}(x) = s_i + j$ where $i = h_k(x), j = h_{k_j}(x)$

  \[
  \begin{array}{|c|c|}
  \hline
  x & \text{pos}(x) \\
  \hline
  2 & 24 \\
  4 & 10 \\
  7 & 3 \\
  11 & 17 \\
  12 & 21 \\
  18 & 19 \\
  19 & 28 \\
  21 & 2 \\
  26 & 22 \\
  28 & 8 \\
  \hline
  \end{array}
  \]