Algorithm Theory

Exercise 1 (Polynomial Coefficients) [Points: 5]
Evaluating a polynomial \( a(x) \) of degree-bound \( n \) at a given point \( x_0 \) can also be done by dividing \( a(x) \) by the polynomial \( (x - x_0) \) to obtain a quotient polynomial \( q(x) \) of degree-bound \( n - 1 \) and a remainder \( r \), such that
\[
a(x) = q(x) \cdot (x - x_0) + r
\]
Clearly, \( a(x_0) = r \). Show how to compute the remainder \( r \) and the coefficients of \( q(x) \) in time \( \Theta(n) \) from \( x_0 \) and the coefficients of \( a \).

Exercise 2 (Interpolation) [Points: 5]
Interpolate the point-value representation
\[
(1, 6), (i, 15 + 15i), (-1, -36), (-i, 15 - 15i)
\]
with FFT to create the coefficient representation of the polynomial.

Exercise 3 (Fast Fourier Transform) [Points: 5]
Compute the product of the two polynomials
\[
p(x) = 4 + 5x \text{ and } q(x) = 9 + 2x
\]
using FFT and interpolation.

Exercise 4 (Fast Fourier Transform) [Points: 5]
Let \( A \) and \( B \) be two sets of integers in the range of \([0, m - 1]\) where \( m \) is a power of two. Show that the following can be computed in \( O(m \cdot \log m) \) time with a single DFT:
1. All elements contained in the set \( A + B = \{c | a \in A, b \in B, c = a + b\} \)
2. For each \( c \in [0, \ldots, 2m - 2] \), the number \( k_c = |\{a, b\} \in A \times B |a + b = c\}|.
**Hint:** Find some polynomials \( p_A, p_B \) of degree less than \( m \) that represent the sets \( A \) and \( B \).