Algorithms Theory

13 – Bin Packing

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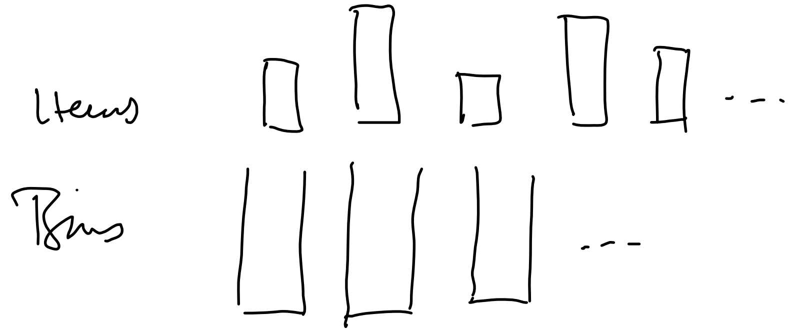
Bin packing

1. Problem definition and general observations

2. Approximation algorithms for the online bin packing problem

3. Approximation algorithms for the offline bin packing problem

Problem definition



Given:

n items with sizes

$$s_1, \dots, s_n$$

where $0 < s_i \leq 1$ for $1 \leq i \leq n$.

Goal:

Pack items into a minimum number of unit-capacity bins.

Example:

7 items with sizes 0.2, 0.5, 0.4, 0.7, 0.1, 0.3, 0.8

Problem definition

Online bin packing:

Items arrive one by one. Each item must be assigned immediately to a bin, without knowledge of any future items. Reassignment is not allowed.

NEXT FIT

FIRST FIT

Offline bin packing:

All n items are known in advance, i.e. before they have to be packed.

FIRST FIT DECREASING

Observations

- Bin packing is provably hard.
  (Offline bin packing is NP-hard.
  Decision problem is NP-complete.)

- There exists no online bin packing algorithm that always finds an optimal solution.

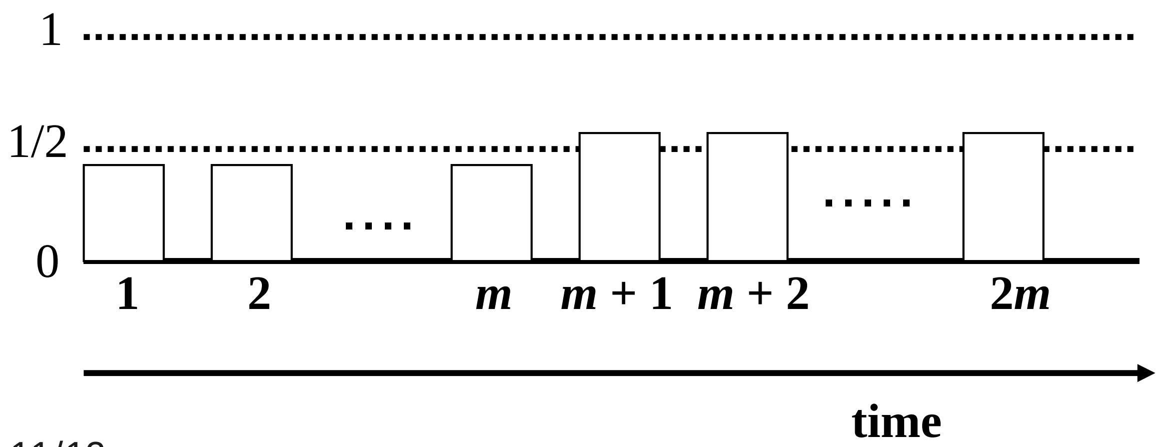
Online bin packing

Theorem 1:

There are inputs that force each online bin packing algorithm to use at least *4/3 OPT* bins where *OPT* is the minimum number of bins possible.

Proof:

Assumption: online bin packing algorithm *A* always uses less than *4/3 OPT* bins



Online bin packing

1st point of time:

$OPT = m/2$ and #bins(A) = b

by assumption: $b < 4/3 \cdot m/2 = 2/3\, m$

Let $b = b_1 + b_2$, with

b_1 = #bins containing one item

b_2 = #bins containing two items

There is: $b_1 + 2\, b_2 = m$, i.e. $b_1 = m - 2b_2$

Hence: $b = b_1 + b_2 = m - b_2$ $(*)$

Online bin packing

2nd point of time:

$OPT = m$

#bins(A) $\geq$ $b + m - b_1$ $=$ $m + b_2$

Assumption: $m + b_2 \leq$ #bins(A) $<$ $4/3m$

$b_2 < m/3$

$\Longrightarrow$ using $(*)$: $b = m - b_2 > 2/3m$

Online bin packing

Next Fit (NF), First Fit (FF), Best Fit (BF)

Next Fit:

Assign an arriving item to the same bin as the preceding item. If it does not fit, open a new bin and place it there.

Theorem 2:

(a) For all input sequences I :

$$NF(I) \leq 2\, OPT(I).$$

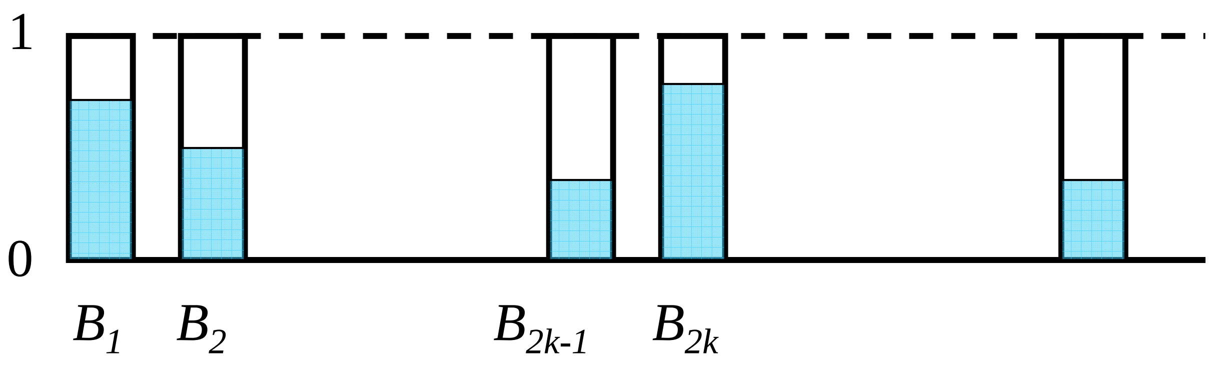
(b) There exist input sequences I such that:

$$NF(I) \geq 2\, OPT(I) - 2.$$

Next Fit

Proof: (a)

Consider two bins B_{2k-1}, B_{2k}, $2k \leq NF(\, I\,)$.



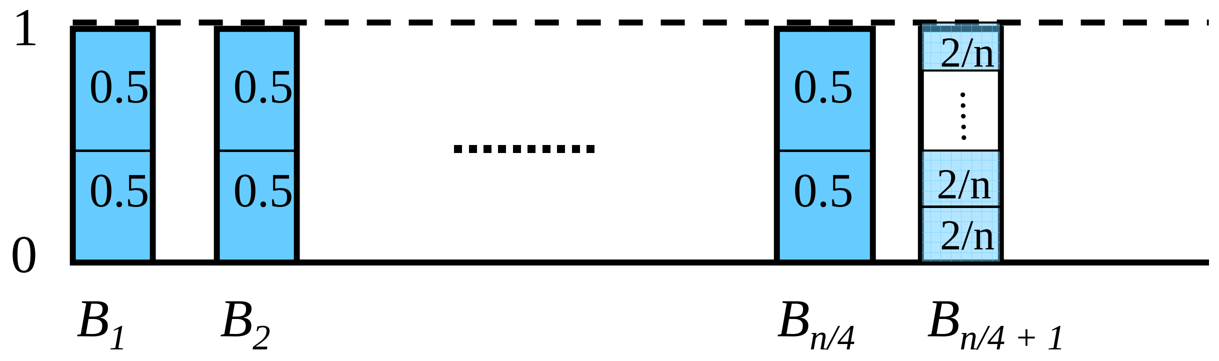
Next Fit

Proof: (b)

Consider an input sequence I of length n
($n \equiv 0 \pmod 4$):

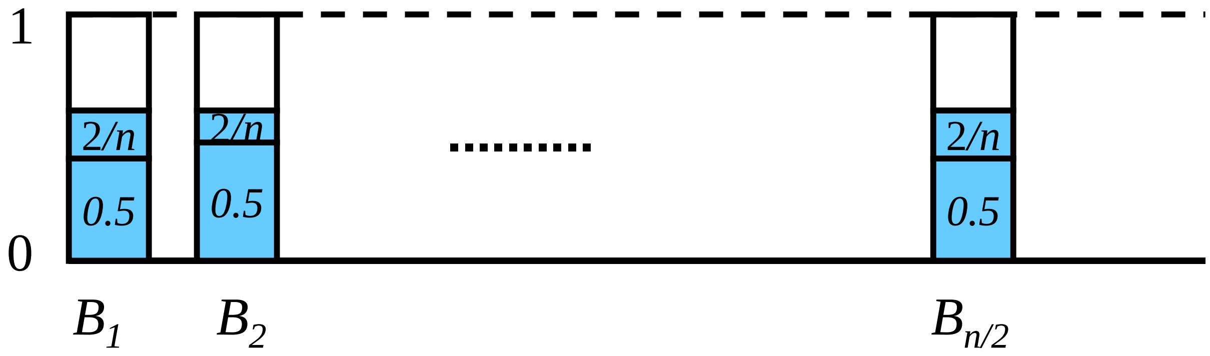
0.5, 2/n, 0.5, 2/n, 0.5, ... , 0.5, 2/n

Optimal packing:



Next Fit

Next Fit yields:



$NF(I) =$

$OPT(I) =$

First Fit

First Fit:

Assign an arriving item to the first bin (i.e. that was opened earliest) in which it fits. If there is no such bin, open a new one and place it there.

Observation:

At each point in time there is at most one bin that is less than half full.

→ $FF(I) \leq 2OPT(I)$

First Fit

Theorem 3:

(a) For all input sequences I :

$$FF(I) \leq \left\lceil \frac{17}{10} OPT(I) \right\rceil$$

involved

(b) There exist input sequences I such that:

$$FF(I) \geq \frac{17}{10} (OPT(I) - 1)$$

(b´) There exist input sequences I such that:

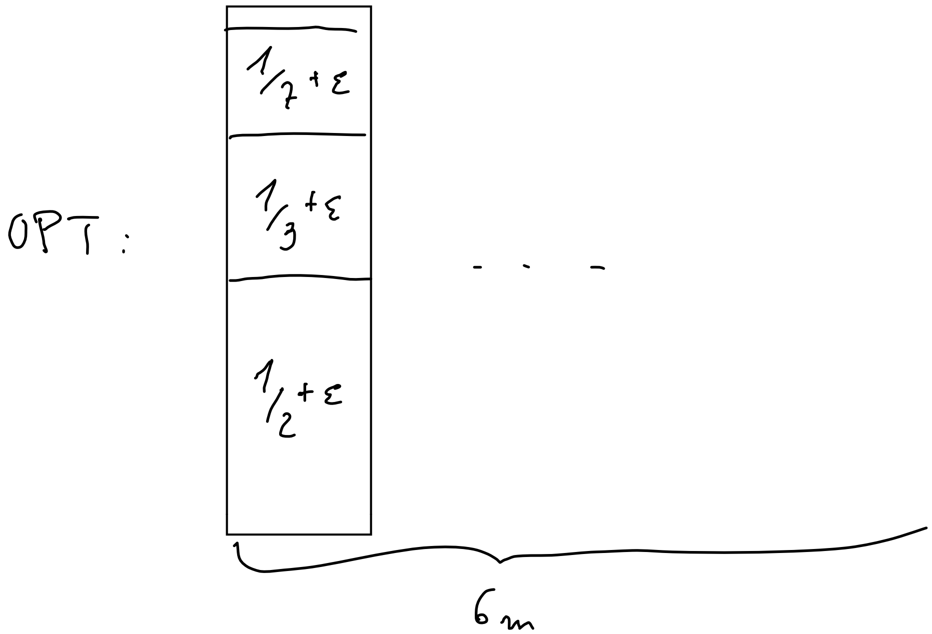
$$FF(I) = \frac{10}{6} OPT(I)$$

First Fit

Proof (b`): Input sequence of length $3 \cdot 6m$:

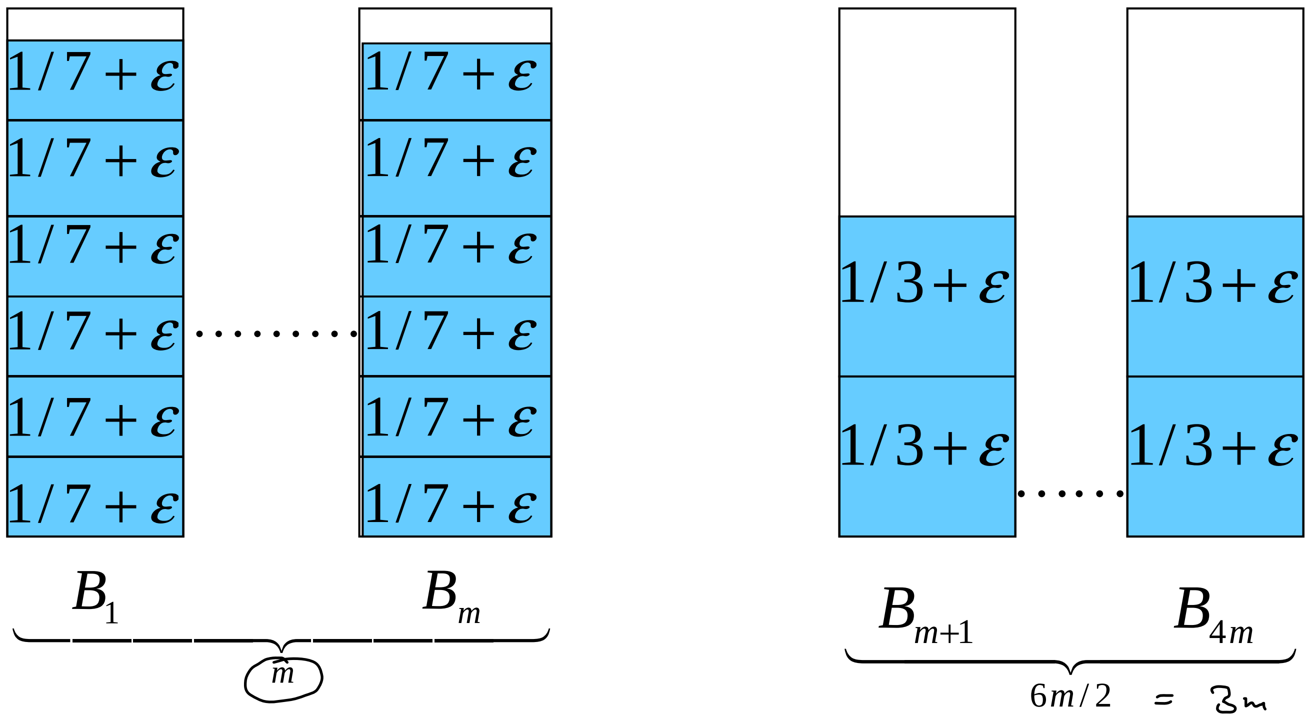
$$\underbrace{1/7+\varepsilon,\ldots,1/7+\varepsilon}_{6m},\underbrace{1/3+\varepsilon,\ldots,1/3+\varepsilon}_{6m},$$

$$\underbrace{1/2+\varepsilon,\ldots,1/2+\varepsilon}_{6m}$$

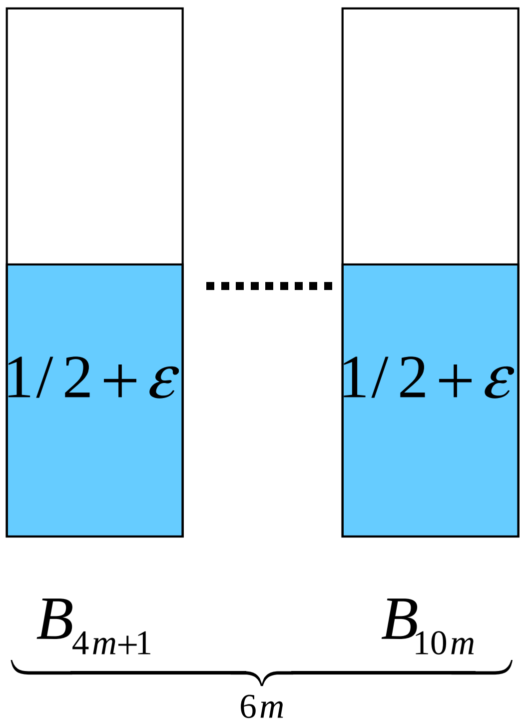


First Fit

First Fit yields:



First Fit



OPT = $6m$

FF = $m + 3m + 6m = 10m$

Best Fit

Best Fit:

Assign an arriving item to the bin in which it fits best (i.e. where it leaves the smallest empty space).

Performance of BF and FF is similar.

Running times on input sequences of length n :

NF $O(n)$

FF $O(n^2)$ $\longrightarrow$ $O(n \log n)$

BF $O(n^2)$ $\longrightarrow$ $O(n \log n)$

naive

data structures

Offline bin packing

Prior to the packing, n and $s_1, \ldots, s_n$ are known in advance.

An optimal packing can be found by exhaustive search. exponential time

Approach to an offline approximation algorithm:

Initially sort the items in decreasing order of size and assign the larger items first!

First Fit Decreasing (FFD) resp. **FFNI**

Best Fit Decreasing (BFD)

First Fit Decreasing

Theorem $FFD(I) \leq \frac{4}{3} \cdot OPT(I) + \frac{1}{3}$

Lemma 1:

Let I be an input sequence of n objects with sizes

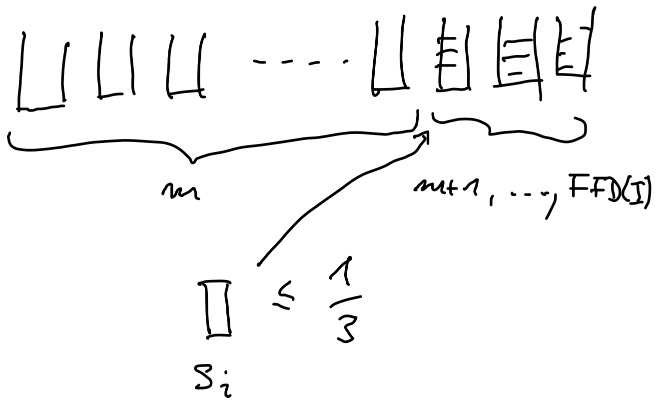
$$s_1 \geq s_2 \geq \ldots\ldots \geq s_n$$

and let $m = OPT(I)$.

Then, all items placed by FFD into bins

$$B_{m+1}, B_{m+2}, \ldots , B_{FFD(I)}$$

are of size at most 1/3.



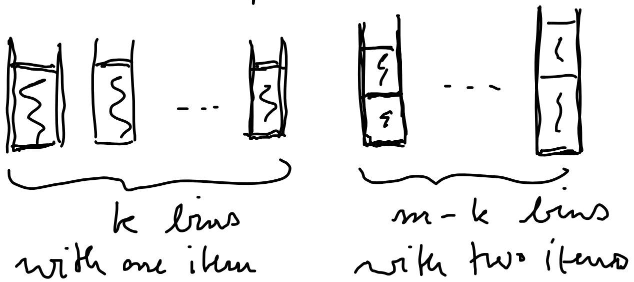
First Fit Decreasing

Proof: Let i be the first item which is placed into bin $m+1$.
We will show $s_i \le 1/3$ implying $s_{i+1}, \dots, s_n \le 1/3$.

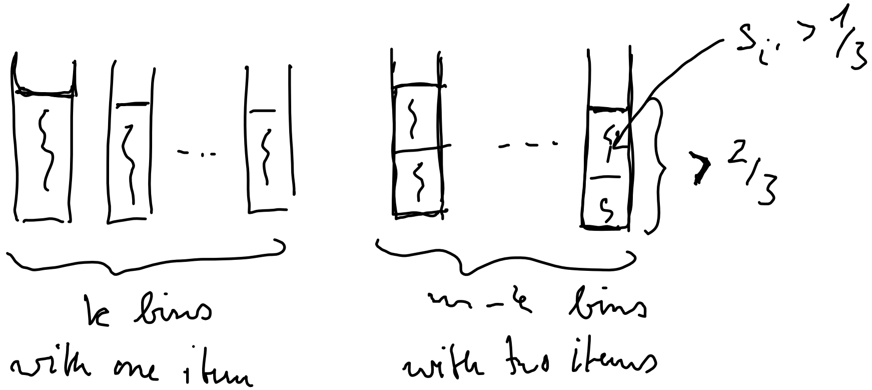
Assume: $s_i > 1/3$

When FFD places i into bin $m+1$ the first m bins have the following configuration

FFD:



OPT:



Item i can not be packed into any of the first k bins (FFD has proved that this is not possible).
And item i can not be packed into any of the last $m-k$ bins because these bins are packed to an extend of $> 2/3$ and the size $s_i > 1/3$.

$OPT(I) > m$ ↯

First Fit Decreasing

Lemma 2:

Let I be an input sequence of n objects with sizes

$$s_1 \geq s_2 \geq \ldots\ldots \geq s_n$$

and let $m = OPT(I)$.

Then the number of items placed by FFD into bins

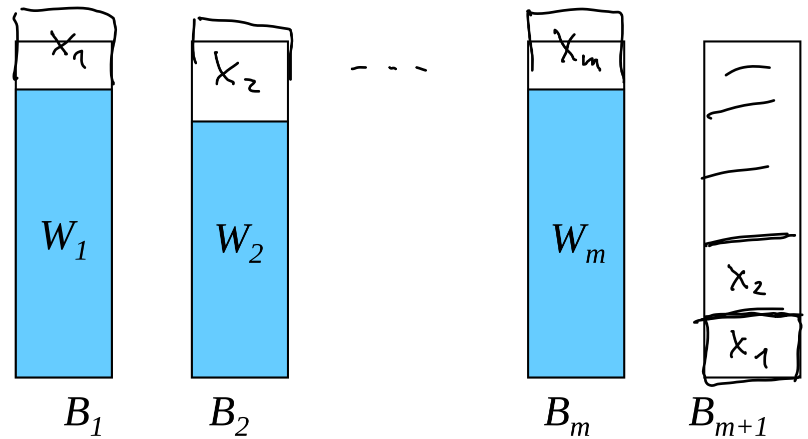
$$B_{m+1}, B_{m+2}, \ldots, B_{FFD(I)}$$

is at most $m - 1$.

First Fit Decreasing

Proof:

Assumption: FFD places more than $m - 1$ items, say $x_1, \ldots, x_m$, into extra bins.



$w_1 + x_1 > 1 \qquad w_2 + x_2 > 1 \qquad \ldots \qquad w_m + x_m > 1$

$$OPT \geq \sum_{i=1}^{n} s_i > \sum_{j=1}^{m} (w_j + x_j) > m \quad ↯$$

First Fit Decreasing

$OPT = m$

Lemma 1+2:

At most $m-1$ items each having size at most $1/3$ are packed into extra bins by FFD.

Thus at most $\lceil \frac{m-1}{3} \rceil$ extra bins

Theorem:

For all input sequences I :

$$FFD(I) \leq (4\, OPT(I) + 1) / 3. = \frac{4}{3} \cdot OPT(I) + \frac{1}{3}$$

Proof $FFD(I) \leq m + \lceil \frac{m-1}{3} \rceil \leq m + \frac{m-1}{3} + \frac{2}{3} = \frac{4}{3} \cdot m + \frac{1}{3} = \frac{4}{3} \cdot OPT(I) + \frac{1}{3}$ ∎

Theorem:

1. For all input sequences I :

$$FFD(I) \leq 11/9\, OPT(I) + 4.$$

difficult

2. There exist input sequences I such that:

$$FFD(I) = 11/9\, OPT(I).$$

First Fit Decreasing

Proof (b): Input sequence of length $3 \cdot 6m + 12m$:

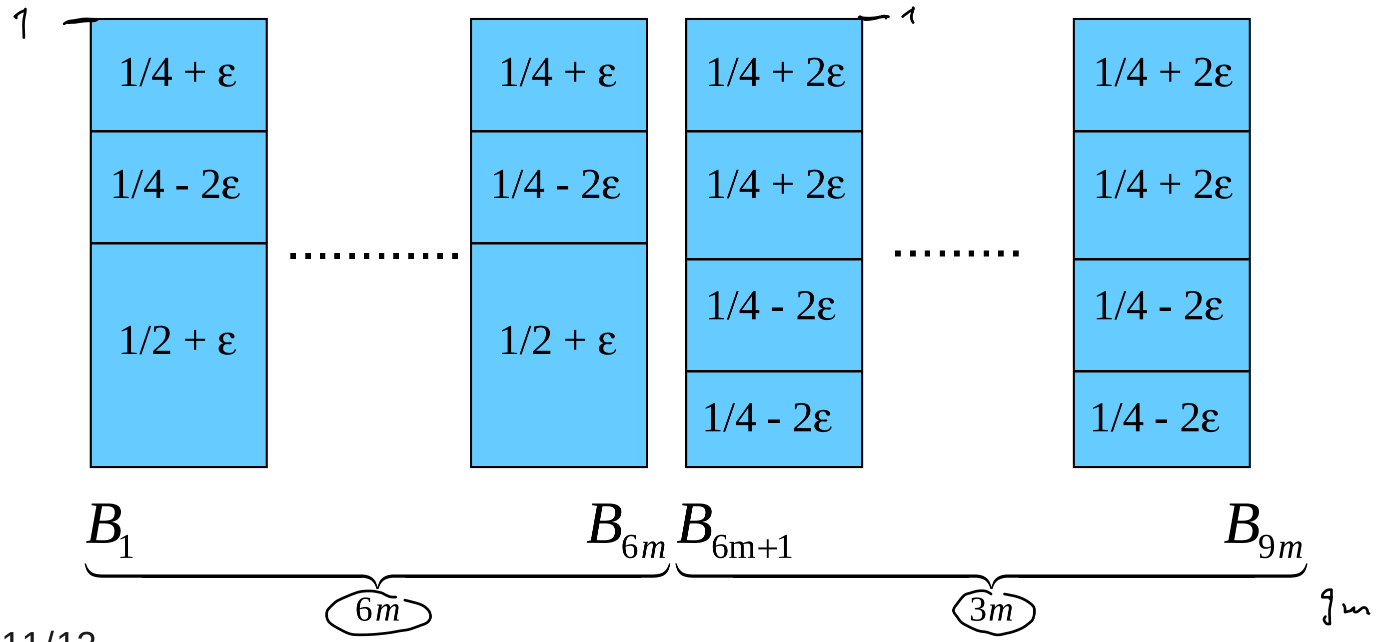
4 types

$$\underbrace{1/2+\varepsilon,\ldots,1/2+\varepsilon}_{6m},\underbrace{1/4+2\varepsilon,\ldots,1/4+2\varepsilon}_{6m}$$

$$\underbrace{1/4+\varepsilon,\ldots,1/4+\varepsilon}_{6m},\underbrace{1/4-2\varepsilon,\ldots,1/4-2\varepsilon}_{12m}$$

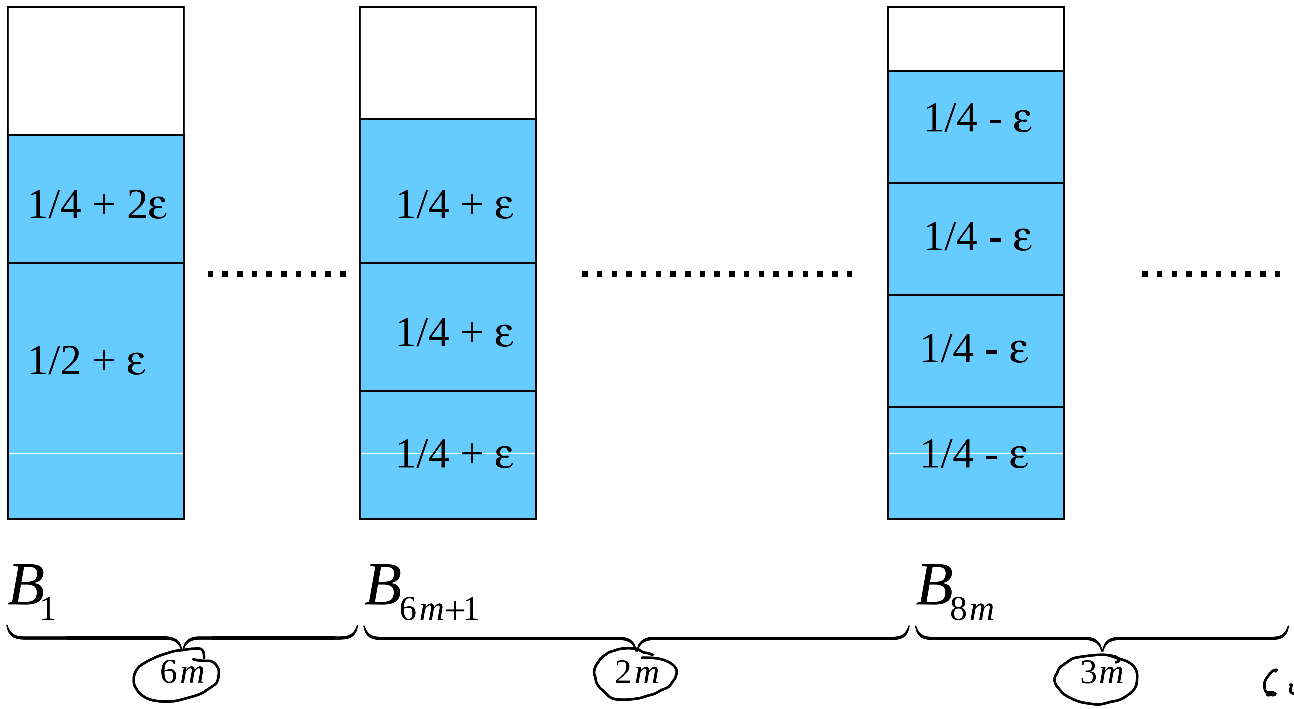
in decreasing order of size

Optimal packing:



First Fit Decreasing

First Fit Decreasing yields:



$6m + 2m + 3m = 11m$

$$OPT(I) = 9m$$
$$FFD(I) = 11m$$