

# Lower Bounds on Information Dissemination in Dynamic Networks

Bernhard Haeupler<sup>1</sup> and Fabian Kuhn<sup>2</sup>

<sup>1</sup> Computer Science and Artificial Intelligence Lab, MIT, USA  
haeupler@mit.edu

<sup>2</sup> Dept. of Computer Science, University of Freiburg, Germany  
kuhn@cs.uni-freiburg.de

**Abstract.** We study lower bounds on information dissemination in adversarial dynamic networks. Initially,  $k$  pieces of information (henceforth called tokens) are distributed among  $n$  nodes. The tokens need to be broadcast to all nodes through a synchronous network in which the topology can change arbitrarily from round to round provided that some connectivity requirements are satisfied.

If the network is guaranteed to be connected in every round and each node can broadcast a single token per round to its neighbors, there is a simple token dissemination algorithm that manages to deliver all  $k$  tokens to all the nodes in  $O(nk)$  rounds. Interestingly, in a recent paper, Dutta et al. proved an almost matching  $\Omega(n + nk/\log n)$  lower bound for deterministic token-forwarding algorithms that are not allowed to combine, split, or change tokens in any way. In the present paper, we extend this bound in different ways.

If nodes are allowed to forward  $b \leq k$  tokens instead of only one token in every round, a straight-forward extension of the  $O(nk)$  algorithm disseminates all  $k$  tokens in time  $O(nk/b)$ . We show that for any randomized token-forwarding algorithm,  $\Omega(n + nk/(b^2 \log n \log \log n))$  rounds are necessary. If nodes can only send a single token per round, but we are guaranteed that the network graph is  $c$ -vertex connected in every round, we show a lower bound of  $\Omega(nk/(c \log^{3/2} n))$ , which almost matches the currently best  $O(nk/c)$  upper bound. Further, if the network is  $T$ -interval connected, a notion that captures connection stability over time, we prove that  $\Omega(n + nk/(T^2 \log n))$  rounds are needed. The best known upper bound in this case manages to solve the problem in  $O(n + nk/T)$  rounds. Finally, we show that even if each node only needs to obtain a  $\delta$ -fraction of all the tokens for some  $\delta \in [0, 1]$ ,  $\Omega(nk\delta^3/\log n)$  are still required.

## 1 Introduction

The growing abundance of (mobile) computation and communication devices creates a rich potential for novel distributed systems and applications. Unlike classical networks, often the resulting networks and applications are characterized by a high level of churn and, especially in the case of mobile devices, a potentially constantly changing topology. Traditionally, changes in a network have been studied as faults or as exceptional events that have to be tolerated and possibly repaired. However, particularly in mobile applications, dynamic networks are a typical case and distributed algorithms have to properly work even under the assumption that the topology is constantly changing.

Consequently, in the last few years, there has been an increasing interest in distributed algorithms that run in dynamic systems. Specifically, a number of recent papers investigate the complexity of solving fundamental distributed computations and information dissemination tasks in dynamic networks, e.g., [2–5, 8, 9, 19, 11, 16–18]. Particularly important in the context of this paper is the synchronous, adversarial dynamic network model defined in [16]. While the network consists of a fixed set of participants  $V$ , the topology can change arbitrarily from round to round, subject to the restriction that the network of each round needs to be connected or satisfy some stronger connectivity requirement.

We study lower bounds on the problem of disseminating a bunch of tokens (messages) to all the nodes in a dynamic network as defined in [16].<sup>1</sup> Initially  $k$  tokens are placed at some nodes in the network. Time is divided into synchronous rounds, the network graph of every round is connected, and in every round, each node can broadcast one token to all its neighbors. If in addition, all nodes know the size of the network  $n$ , we can use the following basic protocol to broadcast all  $k$  tokens to all the nodes. The tokens are broadcast one after the other such that for each token during  $n - 1$  rounds, every node that knows about the token forwards it. Because in each round, there has to be an edge between the nodes knowing the token and the nodes not knowing it, at least one new node receives the token in every round and thus, after  $n - 1$  rounds, all nodes know the token. Assuming that only one token can be broadcast in a single message, the algorithm requires  $k(n - 1)$  rounds to disseminate all  $k$  tokens to all the nodes.

Even though the described approach seems almost trivial, as long as we do not consider protocols based on network coding,  $O(nk)$  is the best upper bound known.<sup>2</sup> In [16], a *token-forwarding algorithm* is defined as an algorithm that needs to forward tokens as they are and is not allowed to combine or change tokens in any way. Note that the algorithm above is a token-forwarding algorithm. In a recent paper, Dutta et al. show that for deterministic token-forwarding algorithms, the described simple strategy indeed cannot be significantly improved by showing a lower bound of  $\Omega(nk/\log n)$  rounds [9]. Their lower bound is based on the following observation. Assume that initially, every node receives every token for free with probability  $1/2$  (independently for all nodes and tokens). Now, with high probability, whatever tokens the nodes decide to broadcast in the next round, the adversary can always find a graph in which new tokens are learned across at most  $O(\log n)$  edges. Hence, in each round, at most  $O(\log n)$  tokens are learned. Because also after randomly assigning tokens with probability  $1/2$ , overall still roughly  $nk/2$  tokens are missing, the lower bound follows. We extend the lower bound from [9] in various natural directions. Specifically, we make the contributions listed in the following. All our lower bounds hold for deterministic algorithms and for randomized algorithms assuming a strongly adaptive adversary (cf. Section 3). Our results are also summarized in Table 1 which is discussed in Section 2.

**Multiple Tokens Per Round:** Assume that instead of forwarding a single token per round, each node is allowed to forward up to  $1 < b \leq k$  tokens in each round. In

<sup>1</sup> To be in line with [16] and other previous work, we refer to the information pieces to be disseminated in the network as tokens.

<sup>2</sup> In fact, if tokens and thus also messages are restricted to a polylogarithmic number of bits, even network coding does not seem to yield more than a polylog. improvement [10, 11].

the simple token-forwarding algorithm that we described above, we can then forward a block of  $b$  tokens to every node in  $n - 1$  rounds and we therefore get an  $O(\frac{nk}{b})$  round upper bound. We show that every (randomized) token-forwarding algorithm needs at least  $\Omega(n + \frac{nk}{b^2 \log n \log \log n})$  rounds.

**Interval Connectivity:** It is natural to assume that a dynamic network cannot change arbitrarily from round to round and that some paths remain stable for a while. This is formally captured by the notion of interval connectivity as defined in [16]. A network is called  $T$ -interval connected for an integer parameter  $T \geq 1$  if for any  $T$  consecutive rounds, there is a stable connected subgraph. It is shown in [16] that in a  $T$ -interval connected dynamic network,  $k$ -token dissemination can be solved in  $O(n + \frac{nk}{T})$  rounds. In this paper, we show that every (randomized) token-forwarding algorithm needs at least  $\Omega(n + \frac{nk}{T^2 \log n})$  rounds.

**Vertex Connectivity:** If instead of merely requiring that the network is connected in every round, we assume that the network is  $c$ -vertex connected in every round for some  $c > 1$ , we can also obtain a speed-up. Because in a  $c$ -vertex connected graph, every vertex cut has size at least  $c$ , if in a round all nodes that know a token  $t$  broadcast it, at least  $c$  new nodes are reached. The basic token-forwarding algorithm thus leads to an  $O(\frac{nk}{c})$  upper bound. We prove this upper bound tight up to a small factor by showing an  $\Omega(\frac{nk}{c \log^{3/2} n})$  lower bound.

**$\delta$ -Partial Token Dissemination:** Finally we consider the basic model, but relax the requirement on the problem by requiring that every node needs to obtain only a  $\delta$ -fraction of all the  $k$  tokens for some parameter  $\delta \in [0, 1]$ . We show that even then, at least  $\Omega(\frac{nk\delta^3}{\log n})$  rounds are needed. This also has implications for algorithms that use forward error correcting codes (FEC) to forward coded packets instead of tokens. We show that such algorithms still need at least  $\Omega(n + k(\frac{n}{\log n})^{1/3})$  rounds until every node has received enough coded packets to decode all  $k$  tokens. Due to lack of space, the discussion of partial token dissemination, as well as some of the easier proofs for the other cases are deferred to the full version of the paper.

## 2 Related Work

As stated in the introduction, we use the network model introduced in [16]. That paper studies the complexity of computing basic functions such as counting the number of nodes in the network, as well as the cost the token dissemination problem that we investigate in the present paper. Previously, some basic results of the same kind were also obtained in [19] for a similar network model.

The token dissemination problem as studied here is first considered in [16] in a dynamic network setting. The paper gives a variant of the distributed  $O(nk)$  token-forwarding algorithm for the case when the number of nodes  $n$  is not known. It is also shown that  $T$ -interval connectivity and always  $c$ -vertex connectivity are interesting parameters that speed up the solution by factors of  $\Theta(T)$  and  $\Theta(c)$ , respectively. In addition, [16] gives a first  $\Omega(n \log k)$  lower bound for token-forwarding algorithms in the centralized setting we study in the present paper. That lower bound is substantially

**Table 1.** Upper and lower bounds for token forwarding (TF) algorithms and network coding (NC) based solutions (bounds in bold are proven in this paper). All TF algorithms are distributed and deterministic while all lower bounds are for centralized randomized algorithms and a strongly adaptive adversary. The NC algorithms work either in the distributed setting against a (standard) adaptive adversary (1) or in the centralized setting against a strongly adaptive adversary (2).

	TF Alg. [16]	NC Alg. [10–12]	TF Lower Bound
always connected	$nk$	$O(\frac{nk}{\log n})$ (1) $O(n+k)$ (2)	$\Omega(n \log k)$ [16] $\Omega(\frac{nk}{\log n})$ [9]
$T$ -interval conn. <sup>+</sup> $T$ -stability*	$\frac{nk}{T}$ (+,*)	$\approx O(n + \frac{nk}{T^2})$ (1,*) $O(n+k)$ (2)	$\Omega(\frac{nk}{T^2 \log n})$ (+)
always $c$ -connected	$\frac{nk}{c}$	$O(\frac{nk}{c \log n})$ (1) $O(\frac{n+k}{c})$ (2)	$\Omega(\frac{nk}{c \log^{3/2} n})$ $\Omega(\frac{nk}{c^2 \log n})$
$b$ -token packets	$\frac{nk}{b}$	$\approx O(\frac{nk}{b^2 \log n})$ (1) $O(n + \frac{k}{b})$ (2)	$\Omega(\frac{nk}{b^2 \log n \log \log n})$
$\delta$ -partial token diss.	$\delta nk$	$O(\frac{\delta nk}{\log n})$ (1) $O(n + \delta k)$ (2)	$\Omega(\frac{\delta^3 nk}{\log n})$

improved in [9], where an almost tight  $\Omega(nk/\log n)$  lower bound is proven. As the lower bound from [9] is the basis of our results, we discuss it in detail in Section 4.1.

The fastest known algorithms for token dissemination in dynamic networks are based on random linear network coding. There, tokens are understood as elements (or vectors) of a finite field and in every round, every node broadcasts a random linear combination of the information it possesses. In a centralized setting, the overhead for transmitting the coefficients of the linear combination can be neglected. For this case, it is shown in [10] that in always connected dynamic networks,  $k$  tokens can be disseminated in optimal  $O(n+k)$  time. If messages are large enough to store  $b$  tokens, this bound improves to again optimal  $O(n+k/b)$  time. It is also possible to extend these results to always  $c$ -connected networks and to the partial token dissemination problem. Note that one possible solution for  $\delta$ -partial token dissemination is to solve regular token dissemination for only  $\delta k$  tokens. If the overhead for coefficients is not neglected, the best known upper bounds are given in [11]. The best bounds for tokens of size  $O(\log n)$ , as well as the upper and lower bounds for the other scenarios are listed in Table 1. The given bound for always  $c$ -vertex connected networks is not proven in [11], it can however be obtained with similar techniques. Note also that instead of  $T$ -interval connectivity, [11] considers a somewhat stronger assumption called  $T$ -stability. In a  $T$ -stable network, the network remains fixed for intervals of length  $T$ .

Apart from token dissemination and basic aggregation tasks, other problems have been considered in the same or similar adversarial dynamic network models. In [17], the problem of coordinating actions in time is studied for always connected dynamic networks. In a recent paper, bounds on what can be achieved if the network is not always connected are discussed in [8]. For a model where nodes know their neighbors before communicating, [2] studies the time to do a random walk if the network can change adversarially. Further, the problem of gradient clock synchronization has been studied for an asynchronous variant of the model [14]. In addition, a number of papers

investigate a radio network variant of essentially the dynamic network model studied here [1, 5, 15]. Another line of research looks at random dynamic networks that result from some Markov process, e.g., [3, 6, 7]. Mostly these papers analyze the time required to broadcast a single message in the network. For a more thorough discussion of related work, we refer to a recent survey [18].

### 3 Model and Problem Definition

In this section we introduce the dynamic network model and the token dissemination problem.

**Dynamic Networks:** We follow the dynamic network model of [16]: A dynamic network consists of a fixed set  $V$  of  $n$  nodes and a dynamic edge set  $E : \mathbb{N} \rightarrow 2^{\{\{u,v\} \mid u,v \in V\}}$ . Time is divided into synchronous rounds so that the network graph of round  $r \geq 1$  is  $G(r) = (V, E(r))$ . We use the common assumption that round  $r$  starts at time  $r - 1$  and it ends at time  $r$ . In each round  $r$ , every node  $v \in V$  can send a message to all its neighbors in  $G(r)$ . Note that we assume that  $v$  has to send the same message to all neighbors, i.e., communication is by local broadcast. Also, we assume that at the beginning of a round  $r$ , when the messages are chosen, nodes are not aware of their neighborhood in  $G(r)$ . We typically assume that the message size is bounded by the size of a fixed number of tokens.

We say that a dynamic network  $G = (V, E)$  is *always  $c$ -vertex connected* iff  $G(r)$  is  $c$ -vertex connected for every round  $r$ . If a network  $G$  is always 1-vertex connected, we also say that  $G$  is *always connected*. Further, we use the definition for interval connectivity from [16]. A dynamic network is  $T$ -interval connected for an integer parameter  $T \geq 1$  iff the graph  $(V, \bigcap_{r'=r}^{r+T-1} E(r'))$  is connected for every  $r \geq 1$ . Hence, a graph is  $T$ -interval connected iff there is a stable connected subgraph for every  $T$  consecutive rounds. Note we do not assume that nodes know the stable subgraph. Also note that a dynamic graph is 1-interval connected iff it is always connected.

For our lower bound, we assume randomized algorithms and a *strongly adaptive adversary* which can decide on the network  $G(r)$  of round  $r$  based on the complete history of the network up to time  $r - 1$  as well as on the messages the nodes send in round  $r$ . Note that the adversary is stronger than the more typical *adaptive adversary* where the graph  $G(r)$  of round  $r$  is independent of the random choices that the nodes make in round  $r$ .

**The Token Dissemination Problem:** We prove lower bounds on the following *token dissemination problem*. There are  $k$  tokens initially distributed among the nodes in the network (for simplicity, we assume that  $k$  is at most polynomial in  $n$ ). We consider *token-forwarding algorithms* as defined in [16]. In each round, every node is allowed to broadcast  $b \geq 1$  of the tokens it knows to all neighbors. Except for Section 4.2, we assume that  $b = 1$ . No other information about tokens can be sent, so that a node  $u$  knows exactly the tokens  $u$  kept initially and the tokens that were included in some message  $u$  received. In addition, we also consider the  $\delta$ -*partial token dissemination* problem. Again, there are  $k$  tokens that are initially distributed among the nodes in the network. But here, the requirement is weaker and we only demand that in the end, every node knows a  $\delta$ -fraction of the  $k$  tokens for some  $\delta \in (0, 1]$ .

We prove our lower bounds for centralized algorithms where a central scheduler can determine the messages sent by each node in a round  $r$  based on the initial state of all the nodes before round  $r$ . Note that lower bounds obtained for such centralized algorithms are stronger than lower bounds for distributed protocols where the message broadcast by a node  $u$  in round  $r$  only depends on the initial state of  $u$  before round  $r$ .

## 4 Lower Bounds

### 4.1 General Technique and Basic Lower Bound Proof

We start our description of the lower bound by outlining the basic techniques and by giving a slightly polished version of the lower bound proof by Dutta et al. [9]. For the discussion here, we assume that in each round, each node is allowed to broadcast a single token, i.e.,  $b = 1$ .

In the following, we make the standard assumption that round  $r$  lasts from time  $r - 1$  to time  $r$ . For each node, we maintain two sets of tokens. For a time  $t \geq 0$  and a node  $u$ , let  $K_u(t)$  be the set of tokens known by node  $u$  at time  $t$ . In addition the adversary determines a token set  $K'_u(t)$  for every node, where  $K'_u(t) \subseteq K'_u(t + 1)$  for all  $t \geq 0$ . The sets  $K'_u(t)$  are constructed such that under the assumption that each node  $u$  knows the tokens  $K_u(t) \cup K'_u(t)$  at time  $t$ , in round  $t + 1$ , overall the nodes cannot learn many new tokens. Specifically, we define a potential function  $\Phi(t)$  as follows:

$$\Phi(t) := \sum_{u \in V} |K_u(t) \cup K'_u(t)|. \quad (1)$$

Note that for the token dissemination problem to be completed at time  $T$  it is necessary that  $\Phi(T) = nk$ . Assume that at the beginning, the nodes know at most  $k/2$  tokens on average, i.e.,  $\sum_{u \in V} |K_u(0)| \leq nk/2$ . For always connected dynamic graphs, we will show that there exists a way to choose the  $K'$ -sets such that  $\sum_{u \in V} |K'_u(0)| < 0.3nk$  and that for every choice of the algorithm, a simple greedy adversary can ensure that the potential grows by at most  $O(\log n)$  per round. We then have  $\Phi(0) \leq 0.8nk$  and since the potential needs to grow to  $nk$ , we get an  $\frac{0.2nk}{O(\log n)}$  lower bound.

In each round  $r$ , for each node  $u$ , an algorithm can decide on a token to send. We denote the token sent by node  $u$  in round  $r$  by  $i_u(r)$  and we call the collection of pairs  $(u, i_u(r))$  for nodes  $u \in V$ , the token assignment of round  $r$ . Note that because a node can only broadcast a token it knows,  $i_u(r) \in K_u(r - 1)$  needs to hold. However, for most of the analysis, we do not make use of this fact and just consider all the  $k$  possible pairs  $(u, i_u(r))$  for a node  $u$ .

If the graph  $G(r)$  of round  $r$  contains the edge  $\{u, v\}$ ,  $u$  or  $v$  learns a new token if  $i_v(r) \notin K_u(r - 1)$  or if  $i_u(r) \notin K_v(r - 1)$ . Moreover, the edge  $\{u, v\}$  contributes to an increase of the potential function  $\Phi$  in round  $r$  if  $i_v(r) \notin K_u(r - 1) \cup K'_u(r - 1)$  or if  $i_u(r) \notin K_v(r - 1) \cup K'_v(r - 1)$ . We call an edge  $e = \{u, v\}$  *free* in round  $r$  iff the edge does not contribute to the potential difference  $\Phi(r) - \Phi(r - 1)$ . In particular, this implies that an edge is free if

$$(i_u(r) \in K'_v(r - 1) \wedge i_v(r) \in K'_u(r - 1)) \vee (i_u(r) = i_v(r))). \quad (2)$$

To construct the  $K'$ -sets we use the probabilistic method. More specifically, for every token  $i$  and all nodes  $u$ , we independently put  $i \in K'_u(0)$  with probability  $p = 1/4$ . The following lemma shows that then only a small number of non-free edges are required in every graph  $G(r)$ .

**Lemma 1 (adapted from [9]).** *If each set  $K'_u(0)$  contains each token  $i$  independently with probability  $p = 1/4$ , for every round  $r$  and every token assignment  $\{(u, i_u(r))\}$ , the graph  $F(r)$  induced by all free edges in round  $r$  has at most  $O(\log n)$  components with probability at least  $3/4$ .*

*Proof.* Assume that the graph  $F(r)$  has at least  $s$  components for some  $s \geq 1$ .  $F(r)$  then needs to have an independent set of size  $s$ , i.e., there needs to be a set  $S \subseteq V$  of size  $|S| \geq s$  such that for all  $u, v \in S$ , the edge  $\{u, v\}$  is not free in round  $r$ . Using (2) and the fact that  $K'_u(0) \subseteq K'_u(t)$  for all  $u$  and  $t \geq 0$ , an edge  $\{u, v\}$  is free in round  $r$  if  $i_u(r) \in K'_v(0)$  and  $i_v(r) \in K'_u(0)$  or if  $i_u(r) = i_v(r)$ .

To argue that  $s$  is always small we use a union bound over all  $\binom{n}{s} < n^s$  ways to choose a set of  $s$  nodes and all at most  $k^s$  ways to choose the tokens to be sent out by these nodes. Note that since two nodes sending out the same token induce a free edge, all tokens sent out by nodes in  $S$  have to be distinct. Furthermore, for any pair of nodes  $u, v \in S$  there is a probability of exactly  $p^2$  for the edge  $\{u, v\}$  to be free and this probability is independent for any pair  $u', v'$  with  $\{u', v'\} \neq \{u, v\}$  because nodes in  $S$  send distinct tokens. The probability that all  $\binom{s}{2} > s^2/4$  node pairs of  $S$  are non-free is thus exactly  $(1 - p^2)^{\binom{s}{2}} < e^{-p^2 s^2/4}$ . If  $s = 12p^{-2} \ln nk > 4p^{-2}(\ln nk + 2)$  (assuming  $\ln(nk) > 1$ ), the union bound  $(nk)^s e^{-p^2 s^2/4}$  is less than  $1/4$  as desired. This shows that there is a way to choose the sets  $K'_u(0)$  such that the greedy adversary always chooses a topology in which the graph  $F(r)$  induced by all free edges has at most  $2s \leq 24p^{-2} \ln nk = O(\log n)$  components.  $\square$

Based on Lemma 1, the lower bound from [9] now follows almost immediately.

**Theorem 1.** *In an always connected dynamic network with  $k$  tokens in which nodes initially know at most  $k/2$  tokens on average, any centralized token-forwarding algorithm takes at least  $\Omega\left(\frac{nk}{\log n}\right)$  rounds to disseminate all tokens to all nodes.*

## 4.2 Sending Multiple Tokens Per Round

In this section we show that it is possible to extend the lower bound to the case where nodes can send out  $b > 1$  tokens in each round. Note that it is a priori not clear that this can be done as for instance the related  $\Omega(n \log k)$  lower bound of [16] breaks down completely if nodes are allowed to send two instead of one tokens in each round.

In order to prove a lower bound for  $b > 1$ , we generalize the notion of free edges. Let us first consider a token assignment for the case  $b > 1$ . Instead of sending a single token  $i_u(r)$ , each node  $u$  now broadcasts a set  $I_u(r)$  of at most  $b$  tokens in every round  $r$ . Analogously to before, we call the collection of pairs  $(u, I_u(r))$  for  $u \in V$ , the token assignment of round  $r$ . We define the weight of an edge in round  $r$  as the amount the

edge contributes to the potential function growth in round  $r$ . Hence, the weight  $w(e)$  of an edge  $e = \{u, v\}$  is defined as

$$w(e) := |I_v(r) \setminus (K_u(r-1) \cup K'_u(r-1))| + |I_u(r) \setminus (K_v(r-1) \cup K'_v(r-1))|. \quad (3)$$

As before, we call an edge  $e$  with weight  $w(e) = 0$  free. Given the edge weights and the potential function as in Section 4.1, a simple possible strategy of the adversary works as follows. In each round, the adversary connects the nodes using an MST w.r.t. the weights  $w(e)$  for all  $e \in \binom{V}{2}$ . The total increase of the potential function is then upper bounded by the weight of the MST.

For the MST to contain  $\ell$  or more edges of weight at least  $w$ , there needs to be set  $S$  of  $\ell + 1$  nodes such that the weight of every edge  $\{u, v\}$  for  $u, v \in S$  is at least  $w$ . The following lemma bounds the probability for this to happen, assuming that the  $K'$ -sets are chosen randomly such that every token  $i$  is contained in every set  $K'_u(0)$  with probability  $p = 1 - \varepsilon/(4eb)$  for some constant  $\varepsilon > 0$ .

**Lemma 2.** *Assume that each set  $K'_u(0)$  contains each token independently with probability  $1 - \varepsilon/(4eb)$ . Then, for every token assignment  $(u, I_u(r))$ , there exists a set  $S$  of size  $\ell + 1$  such that all edges connecting nodes in  $S$  have weight at least  $w$  with probability at most*

$$\exp\left((\ell + 1) \cdot \left(\ln n + b \ln k + \ell + 1 - \frac{\ell w}{12} \ln\left(\frac{w}{\varepsilon}\right)\right)\right).$$

*Proof.* Consider an arbitrary (but fixed) set of nodes  $v_0, \dots, v_\ell$  and a set of token sets  $T_0, \dots, T_\ell$  (we assume that the token assignment contains the  $\ell + 1$  pairs  $(v_i, T_i)$ ). We define  $\mathcal{E}_i$  to be the event that  $|\bigcup_{j \neq i} T_j \setminus K'_{v_i}(0)| > \ell w/4$ . Note that whenever  $|K_{v_i} \cup K'_{v_i}|$  grows by more than  $\ell w/4$ , the event  $\mathcal{E}_i$  definitely happens. In order to have  $|T_j \setminus K'_{v_i}(0)| + |T_i \setminus K'_{v_j}(0)| \geq w$  for each  $i \neq j$ , at least  $(\ell + 1)/3$  of the events  $\mathcal{E}_i$  need to occur. Hence, for all edges  $\{v_i, v_j\}$ ,  $i, j \in \{0, \dots, \ell\}$ , to have weight at least  $w$ , at least  $(\ell + 1)/3$  of the events  $\mathcal{E}_i$  have to happen. As the event  $\mathcal{E}_i$  only depends on the randomness used to determine  $K'_{v_i}(0)$ , events  $\mathcal{E}_i$  for different  $i$  are independent. The number of events  $\mathcal{E}_i$  that occur is thus dominated by a binomial random variable  $\text{Bin}(\ell + 1, \max_i \mathbb{P}[\mathcal{E}_i])$  variable with parameters  $\ell + 1$  and  $\max_i \mathbb{P}[\mathcal{E}_i]$ . The probability  $\mathbb{P}[\mathcal{E}_i]$  for each  $i$  can be bounded as follows:

$$\mathbb{P}[\mathcal{E}_i] \leq \binom{\ell b}{\ell w/4} \cdot \left(\frac{\varepsilon}{4eb}\right)^{\ell w/4} \leq \left(\frac{4e\ell b}{\ell w}\right)^{\ell w/4} \cdot \left(\frac{\varepsilon}{4eb}\right)^{\ell w/4} = \left(\frac{\varepsilon}{w}\right)^{\ell w/4}.$$

Let  $X$  be the number of events  $\mathcal{E}_i$  that occur. We have

$$\mathbb{P}\left[X \geq \frac{\ell + 1}{3}\right] \leq \binom{\ell + 1}{(\ell + 1)/3} \cdot \left(\frac{\varepsilon}{w}\right)^{\frac{\ell w}{4} \cdot \frac{\ell + 1}{3}} \leq 2^{\ell + 1} \cdot \left(\frac{\varepsilon}{w}\right)^{\frac{\ell w}{4} \cdot \frac{\ell + 1}{3}}.$$

The number of possible ways to choose  $\ell + 1$  nodes and assign a set of  $b$  tokens to each node is

$$\binom{n}{\ell + 1} \cdot \binom{k}{b}^{\ell + 1} \leq (nk^b)^{\ell + 1}.$$



The claim of the lemma now follows by applying a union bound over all possible choices  $v_0, \dots, v_\ell$  and  $T_0, \dots, T_\ell$ .  $\square$

Based on Lemma 2, we obtain the following theorem.

**Theorem 2.** *On always connected dynamic networks with  $k$  tokens in which nodes initially know at most  $k/2$  tokens on average, every centralized randomized token-forwarding algorithm requires at least*

$$\Omega\left(\frac{nk}{(\log n + b \log k)b \log \log b}\right) \geq \Omega\left(\frac{nk}{b^2 \log n \log \log n}\right)$$

*rounds to disseminate all tokens to all nodes.*

*Proof.* For  $w_i = 2^i$ , let  $\ell_i + 1$  be the size of the largest set  $S_i$ , such that that edge between any two nodes  $u, v \in S_i$  has weight at least  $w_i$ . Hence, in the MST, there are at most  $\ell_i$  edges with weight between  $w_i$  and  $2w_i$ . The amount by which the potential function  $\Phi$  increases in round  $r$  can then be upper bounded by

$$\sum_{i=0}^{\log b} 2w_i \cdot \ell_i = \sum_{i=0}^{\log b} 2^{i+1} \cdot \ell_i.$$

By Lemma 2 (and a union bound over the  $\log b$  different  $w_i$ ), for a sufficiently small constant  $\varepsilon > 0$ ,

$$\ell_i = O\left(\frac{\log n + b \log k}{w_i \log w_i}\right) = O\left(\frac{\log n + b \log k}{2^i \cdot i}\right)$$

with high probability. The number of tokens learned in each round can thus be bounded by

$$\sum_{i=0}^{\log b} O\left(\frac{\log n + b \log k}{i}\right) = O((\log n + b \log k) \log \log b).$$

By a standard Chernoff bound, with high probability, the initial potential is of the order  $1 - \Theta(nk/b)$ . Therefore to disseminate all tokens to all nodes, the potential has to increase by  $\Theta(nk/b)$  and the claim follows.  $\square$

### 4.3 Interval Connected Dynamic Networks

While allowing that the network can change arbitrarily from round to round is a clean and useful theoretical model, from a practical point of view it might make sense to look at dynamic graphs that are a bit more stable. In particular, some connections and paths might remain reliable over some period of time. In [16], token dissemination and the other problems considered are studied in the context of  $T$ -interval connected graphs. For  $T$  large enough, sufficiently many paths remain stable for  $T$  rounds so that it is possible to use pipelining along the stable paths to disseminate tokens significantly faster (note that this is possible even though the nodes do not know which edges are stable). In the following, we show that the lower bound described in Section 4.1 can also be extended to  $T$ -interval connected networks.

**Theorem 3.** *On  $T$ -interval connected dynamic networks in which nodes initially know at most  $k/2$  of  $k$  tokens on average, every randomized token-forwarding algorithm requires at least*

$$\Omega\left(\frac{nk}{T(T \log k + \log n)}\right) \geq \Omega\left(\frac{nk}{T^2 \log n}\right)$$

*rounds to disseminate all tokens to all nodes.*

*Proof.* We assume that each of the sets  $K'_u(0)$  independently contains each of the  $k$  tokens with probability  $p = 1 - \varepsilon/T$  for a sufficiently small constant  $\varepsilon > 0$ . As before, we let  $i_u(r)$  be the token broadcast by node  $u$  in round  $r$  and call the set of pairs  $(u, i_u(r))$  the token assignment of round  $r$ . In the analysis, we will also make use of token assignments of the form  $\mathcal{T} = \{(u, I_u) : u \in V\}$ , where  $I_u$  is a set of tokens sent by some node  $u$ .

Given a token assignment  $\mathcal{T} = \{(u, I_u)\}$ , as in the previous subsection, an edge  $\{u, v\}$  is free in particular if  $I_u \subseteq K'_v(0) \wedge I_v \subseteq K'_u(0)$ . Let  $E_{\mathcal{T}}$  be the free edges w.r.t. a given token assignment  $\mathcal{T}$ . Further, we define  $\mathcal{S}_{\mathcal{T}} = \{S_{\mathcal{T},1}, \dots, S_{\mathcal{T},\ell}\}$  to be the partition of  $V$  induced by the components of the graph  $(V, E_{\mathcal{T}})$ .

Consider a sequence of  $2T$  consecutive rounds  $r_1, \dots, r_{2T}$ . For a node  $v_j$  and round  $r_i, i \in [2T]$ , let  $I_{i,j} := \{i_{v_j}(r_1), \dots, i_{v_j}(r_i)\}$  be the set of tokens transmitted by node  $v_j$  in rounds  $r_1, \dots, r_i$  and let  $\mathcal{T}_i := \{(v_1, I_{i,1}), \dots, (v_n, I_{i,n})\}$ . As above, let  $E_{\mathcal{T}_i}$  be the free edges for the token assignment  $\mathcal{T}_i$  and let  $\mathcal{S}_{\mathcal{T}_i}$  be the partition of  $V$  induced by the components of the graph  $(V, E_{\mathcal{T}_i})$ . Note that for  $j > i, E_{\mathcal{T}_j} \subseteq E_{\mathcal{T}_i}$  and  $\mathcal{S}_{\mathcal{T}_j}$  is a sub-division of  $\mathcal{S}_{\mathcal{T}_i}$ .

We construct edge sets  $E_1, \dots, E_{2T}$  as follows. The set  $E_1$  contains  $|\mathcal{S}_{\mathcal{T}_1}| - 1$  edges to connect the components of the graph  $(V, E_{\mathcal{T}_1})$ . For  $i > 1$ , the edge set  $E_i$  is chosen such that  $E_i \subseteq E_{\mathcal{T}_{i-1}}, |E_i| = |\mathcal{S}_{\mathcal{T}_i}| - |\mathcal{S}_{\mathcal{T}_{i-1}}|$ , and the graph  $(V, E_{\mathcal{T}_i} \cup E_1 \cup \dots \cup E_i)$  is connected. Note that such a set  $E_i$  exists by induction on  $i$  and because  $\mathcal{S}_{\mathcal{T}_i}$  is a sub-division of  $\mathcal{S}_{\mathcal{T}_{i-1}}$ .

For convenience, we define  $E_{\{r_1, \dots, r_i\}} := E_1 \cup \dots \cup E_i$ . By the above construction, the number of edges in  $E_{\{r_1, \dots, r_i\}}$  is  $|\mathcal{S}_{\mathcal{T}_i}| - 1$ , where  $|\mathcal{S}_{\mathcal{T}_i}|$  is the number of components of the graph  $(V, E_{\mathcal{T}_i})$ . Because in each round, every node transmits only one token, the number of tokens in each  $I_{i,j} \in \mathcal{T}_i$  is at most  $|I_{i,j}| \leq i \leq 2T$ . By Lemma 2, if the constant  $\varepsilon$  is chosen small enough, the number of components of  $(V, E_{\mathcal{T}})$  and therefore the size of  $E_{\{r_1, \dots, r_i\}}$  is upper bounded by  $|\mathcal{S}_{\mathcal{T}}| \leq \log n + T \log k$ , w.h.p.

We construct the dynamic graph as follows. For simplicity, assume that the first round of the execution is round 0. Consider some round  $r$  and let  $r_0$  be the largest round number such that  $r_0 \leq r$  and  $r_0 \equiv 0 \pmod{T}$ . The edge set in round  $i$  consists of the the free edges in round  $i$ , as well as of the sets  $E_{i_0-T, \dots, i}$  and  $E_{i_0, \dots, i}$ . The resulting dynamic graph is  $T$ -interval-connected. Furthermore, the number of non-free edges in each round is  $O(\log n + T \log k)$ . Because in each round, at most 2 tokens are learned over each non-free edge, the theorem follows.  $\square$

#### 4.4 Vertex Connectivity

Rather than requiring more connectivity over time, we now consider the case when the network is better connected in every round. If the network is  $c$ -vertex connected for

some  $c > 1$ , in every round, each set of nodes can potentially reach  $c$  other nodes (rather than just 1). In [16], it is shown that for the basic greedy token forwarding algorithm, one indeed gains a factor of  $\Theta(c)$  if the network is  $c$ -vertex connected in every round. We first need to state two general facts about vertex connected graphs and a basic result about weighted sums of Bernoulli random variables.

**Proposition 1.** *If in a graph  $G$  there exists a vertex  $v$  with degree at least  $c$  such that  $G - \{v\}$  is  $c$ -vertex connected then  $G$  is also  $c$ -vertex connected.*

By specializing a much more powerful result of [13], we can characterize the minimum number of edges needed to augment a graph to be  $c$ -vertex connected.

**Lemma 3.** *For  $c$ , any  $n$ -node graph  $G = (V, E)$  with minimum degree at least  $2c - 2$  can be augmented by  $n$  edges to be  $c$ -vertex connected.*

**Lemma 4.** *For some  $c$  let  $\ell_1, \ell_2, \dots, \ell_\tau$  be positive integers with  $\ell = \sum_i \ell_i > c$ . Furthermore, let  $X_1, X_2, \dots, X_\tau$  be i.i.d. Bernoulli variables with  $\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = 0] = 1/2$  for all  $i$ . For any integer  $x > 1$  it holds that:*

$$\mathbb{P} \left[ \min \left\{ |L| : L \subseteq [\tau] \wedge \sum_{i \in \{j | X_j = 1\} \cup L} \ell_i \geq c \right\} > x \right] < 2^{-\Theta(\frac{x\ell}{c})}.$$

*That is, the probability that  $x$  of the random variables need to be switched to one after a random assignment in order get  $\sum_i X_i \ell_i \geq c$  is at most  $2^{-\Theta(\frac{x\ell}{c})}$ .*

To prove our lower bound for always  $c$ -connected graph, we initialize the  $K'$ -sets as for always connected graphs, i.e., each token  $i$  is contained in every set  $K'_u(0)$  with constant probability  $p$  (we assume  $p = 1/2$  in the following). In each round, the adversary picks a  $c$ -connected graph with as few free edges as possible. Using Lemmas 1 and 3, we will show that a graph with a small number of non-free edges can be constructed as follows. First, as long as we can, we pick vertices with at least  $c$  neighbors among the remaining nodes. We then show how to extend the resulting graph to a  $c$ -connected graph.

**Lemma 5.** *With high probability (over the choices of the sets  $K'_u(0)$ ), for every token assignment  $(u, I_u(r))$ , the largest set  $S$  for which no node  $u \in S$  has at least  $c$  neighbors in  $S$  is of size  $O(c \log n)$ .*

*Proof.* Consider some round  $r$  with token assignment  $\{(u, i_u(r))\}$ . We need to show that for any set  $S$  of size  $s = \alpha c \log n$  for a sufficiently large constant  $\alpha$ , at least one node in  $S$  has at least  $c$  free neighbors in  $S$  (i.e., the largest degree of the graph induced by the free edges between nodes in  $S$  is at least  $c$ ).

We will use a union bound over all  $n^s$  sets  $S$  and all  $k^s$  possibilities for selecting the tokens sent by these nodes. We want to show that if the constant  $\alpha$  is chosen sufficiently large, for each of these  $2^{s \log nk}$  possibilities we have a success probability of at least  $1 - 2^{-2^s \log nk}$ .

We first partition the nodes in  $S$  according to the token sent out, i.e.,  $S_i$  is the subset of nodes sending out token  $i$ . Note that if for some  $j$  we have  $|S_j| > c$  we are done since all edges between nodes sending the same token are free. With this, let  $j^*$  be such that  $\sum_{i < j^*} |S_i| \geq s/3$  and  $\sum_{i > j^*} |S_i| \geq s/3$ . We now claim that for every  $j < j^*$ , with probability at most  $2^{-6|S_j| \log nk}$ , there does not exist a node in  $S_j$  that has at least  $c$  free edges to nodes in  $S' = \bigcup_{i > j^*} S_i$ . Note that the events that a node from  $S_j$  has at least  $c$  free edges to nodes in  $S'$  are independent for different  $j$  as it only depends on which nodes  $u$  in  $S'$  have  $j$  in  $K'_u(0)$  and on the  $K'(0)$ -sets of the nodes in  $S_j$ . The claim that we have a node with degree  $c$  in  $S$  with probability at least  $1 - 2^{-2s \log nk}$  then follows from the definition of  $j^*$ .

Let us therefore consider a fixed value  $j$ . We first note that for a fixed  $j$  by standard Chernoff bounds with probability at least  $1 - 2^{-\Theta(s)}$ , there are at least  $s/3 \cdot p/2 = s/12$  nodes in  $S'$  that have token  $j$  in their initial  $K'$ -set. For  $\alpha$  sufficiently large, this probability is at least  $1 - 2^{-7c \log nk} \geq 1 - 2^{-7|S_j| \log nk}$ . In the following, we assume that there are at least  $s/12$  nodes  $u$  in  $S'$  for which  $j \in K'_u(0)$ .

Let  $s_{j,i}$  for any  $i > j^*$  denote the number of nodes in  $S_i$  that have token  $j$  in the initial  $K'$ -set. The number of free edges to a node  $u$  in  $S_j$  is at least  $\sum_{i > j^*} X_{u,i} s_{i,j}$ , where the random variable  $X_{u,i}$  is 1 if node  $u$  initially has token  $i$  in  $K'_u(0)$  and 0 otherwise (i.e.,  $X_{u,i}$  is a Bernoulli variable with parameter  $1/2$ ). Note that since  $\sum_i s_{j,i} \geq s/12$ , the expected value of the number of free edges to a node  $u$  in  $S_j$  is at least  $s/24$ . By a Chernoff bound, the probability that the number of free edges from a node  $u$  in  $S_j$  does not deviate by more than a constant factor with probability  $1 - 2^{-\Theta(s/c)}$ . Note that  $s_{j,i} \leq c$  since  $|S_j| \leq c$ . For  $\alpha$  large enough this probability is at least  $1 - 2^{-7 \log nk}$ . Because the probability bound only depends on the choice of  $K'_u(0)$ , we have independence for different  $u \in S_j$ . Therefore, given that at least  $s/12$  nodes in  $S'$  have token  $j$ , the probability that no node in  $S_j$  has at least  $c$  neighbors in  $S'$  can be upper bounded as  $(1 - 2^{-7|S_j| \log nk})$ . Together with the bound on the probability that at least  $s/12$  nodes in  $S'$  have token  $j$  in their  $K'(0)$  set, the claim of the lemma follows.  $\square$

Lemma 5 by itself directly leads to a lower bound for token forwarding algorithms in always  $c$ -vertex connected graphs.

**Corollary 1.** *Suppose an always  $c$ -vertex connected dynamic network with  $k$  tokens in which nodes initially know at most a constant fraction of the tokens on average. Then, any centralized token-forwarding algorithm takes at least  $\Omega\left(\frac{nk}{c^2 \log n}\right)$  rounds to disseminate all tokens to all nodes.*

*Proof.* By Lemma 5, we know that there exists  $K'(0)$ -sets such that for every token assignment after adding all free edges, the size of the largest induced subgraph with maximum degree less than  $c$  is  $O(c \log n)$ . By Lemma 1, it suffices to make the graph induced by these  $O(c \log n)$  nodes  $c$ -vertex connected to have a  $c$ -vertex connected graph on all  $n$  nodes. To achieve this, by Lemma 3, it suffices to increase all degrees to  $2c - 2$  and add another  $O(c \log n)$  edges. Overall, the number of non-free edges we have to add for this is therefore upper bounded by  $O(c^2 \log n)$ . Hence, the potential function increases by at most  $O(c^2 \log n)$  per round and since we can choose the  $K'(0)$ -sets so that initially the potential is at most  $\lambda nk$  for a constant  $\lambda < 1$ , the bound follows.  $\square$

As shown in the following, by using a more careful analysis, we can significantly improve this lower bound for  $c = \omega(\log n)$ . Note that the bound given by the following theorem is at most an  $O(\log^{3/2} n)$  factor away from the simple “greedy” upper bound.

**Theorem 4.** *Suppose an always  $c$ -vertex connected dynamic network with  $k$  tokens in which nodes initially know at most a constant fraction of the tokens on average. Then, any centralized token-forwarding algorithm takes at least  $\Omega\left(\frac{nk}{c \log^{3/2} n}\right)$  rounds to disseminate all tokens to all nodes.*

*Proof.* We use the same construction as in Lemma 5 to obtain a set  $S$  of size  $|S| = s = \alpha c \log n$  for a sufficiently large constant  $\alpha > 0$  such that  $S$  needs to be augmented to a  $c$ -connected graph. Note that we want the set to be of size  $s$  and therefore we do not assume that in the induced subgraph, every node has degree less than  $c$ . We improve upon Lemma 5 by showing that it is possible to increase the potential function by adding a few more tokens to the  $K'$ -sets, so that afterwards it is sufficient to add  $O(s)$  additional non-free edges to  $S$  to make the induced subgraph  $c$ -vertex connected. Hence, an important difference is that are not counting the number of edges that we need to add but the number of tokens we need to give away (i.e., add to the existing  $K'$ -sets).

We first argue that w.h.p., it is possible to raise the minimum degree of vertices in the induced subgraph of  $S$  to  $2c$  without increasing the potential function by too much. Then we invoke Lemma 3 and get that at most  $O(s)$  more edges are then needed to make  $S$  induce a  $c$ -connected graph as desired.

We partition the nodes in  $S$  according to the token sent out in the same way as in the proof of Lemma 5, i.e.,  $S_i$  is the subset of nodes sending out token  $i$ . Let us first assume that no set  $S_i$  contains more than  $s/3$  nodes. We can then divide the sets of the partition into two parts with at least  $s/3$  nodes each. To argue about the sets, we rename the tokens sent out by nodes in  $S$  as  $1, 2, \dots$  so that we can find a token  $j^*$  for which  $\sum_{j=1}^{j^*} |S_j| \geq s/3$  and  $\sum_{j>j^*} |S_j| \geq s/3$ . We call the sets  $S_j$  for  $j \leq j^*$  the left side of  $S$  and the sets  $S_j$  for  $j > j^*$  the right side of  $S$ . If there is a set  $S_i$  with  $|S_i| > s/3$ , we define  $S_i$  to be the right side and all other sets  $S_j$  to be the left side of  $S$ . We will show that we can increase the potential function by at most  $O(s\sqrt{\log n}) = O(c \log^{3/2} n)$  such that all the nodes on the left side have at least  $2c$  neighbors on the right side. If all sets  $S_i$  are of size at most  $s/3$ , increasing the degrees of the nodes on the right side is then done symmetrically. If the right side consists of a single set  $S_i$  of size at least  $s/3$ , for  $\alpha$  large enough we have  $s/3 \geq 2c + 1$  and therefore nodes on the right side already have degree at least  $2c$  by just using free edges.

We start out by adding some tokens to the sets  $K'_u$  for nodes  $u$  on the right side such that for every token  $j \leq j^*$  on the left side, there are at least  $s/\sqrt{\log n}$  nodes  $u$  on the right side for which  $j \in K'_u$ . Let us consider some fixed token  $j \leq j^*$  from the left side. Because every node  $u$  on the right side has  $j \in K'_u(0)$  with probability  $1/2$ , with probability at least  $1 - 2^{-\Theta(s)}$ , at least  $s/\sqrt{\log n}$  nodes  $u$  on the right side have  $j \in K'_u(0)$ . For such a token  $j$ , we do not need to do anything. Note that the events that  $j \in K'_u(0)$  are independent for different  $j$  on the left side. Therefore, for a sufficiently large constant  $\beta$  and a fixed collection of  $\beta \log n$  tokens  $j$  sent by nodes on the left side, the probability that none of these tokens is in at least  $s/\sqrt{\log n}$  sets  $K'_u(0)$  for  $u$  on the

right side is at most  $2^{-\gamma s \log n}$  for a given constant  $\gamma > 0$ . As there are at most  $s$  tokens sent by nodes on the left side, the number of collections of  $\beta \log n$  tokens is at most

$$\binom{s}{\beta \log n} \leq \left(\frac{es}{\beta \log n}\right)^{\beta \log n} = \left(\frac{e\alpha c}{\beta}\right)^{\beta \log n} = 2^{\theta(\log c \log n)},$$

which is less than  $2^{s \log n}$  for sufficiently large  $\alpha$ . Hence, with probability at least  $1 - 2^{-(\gamma-1)s \log n}$ , for at most  $\beta \log n$  tokens  $j$  on the left side there are less than  $s/\sqrt{\log n}$  nodes  $u$  on the right that have  $j \in K'_u(0)$ . For these  $O(\log n)$  tokens  $j$ , we add to  $j$  to  $K'_u$  for at most  $s/\sqrt{\log n}$  nodes  $u$  on the right side, such that afterwards, for every token  $j$  sent by a node on the left side, there are at least  $s/\sqrt{\log n}$  nodes  $u$  on the right for which  $j \in K'_u$ . Note that this increases the potential function by at most  $O(s\sqrt{\log n}) = O(c \log^{3/2} n)$ .

We next show that by adding another  $O(c \log^{3/2} n)$  tokens to the  $K'$ -sets of the nodes on the left side, we manage to get that every node  $u$  on the left side has at least  $2c$  free neighbors on the right side. For a token  $j \leq j^*$  sent by some node on the left side and a token  $i > j^*$  sent by some node on the right side, let  $s_{i,j}$  be the number of nodes  $u \in S_i$  for which  $j \in K'_u$ . Note that if token  $i$  is in  $K'_v$  for some  $v \in S_j$ ,  $v$  has  $s_{i,j}$  neighbors in  $S_i$ .

Using the augmentation of the  $K'_u$ -sets for nodes on the right, we have that for every  $j \leq j^*$ ,  $\sum_{i > j^*} s_{i,j} \geq s/\sqrt{\log n}$ . For every  $i > j^*$ , with probability  $1/2$ , we have  $i \in K'_v(0)$ . In addition, we add tokens additional  $i$  to  $K'_v$  for which  $i \notin K'_v(0)$  such that in the end,  $\sum_{i > j^*, i \in K'_v} s_{i,j} \geq 2c$ . By Lemma 4, the probability that we need to add  $\geq x$  tokens is upper bounded by  $2^{-\theta(xs/(c\sqrt{\log n}))} = 2^{-\theta(x\sqrt{\log n})}$ . As the number of tokens we need to add to  $K'_v$  is independent for different  $v$ , in total we need to add at most  $O\left(\frac{s \log n}{\sqrt{\log n}}\right) = O(c \log^{3/2} n)$  tokens with probability at least  $1 - 2^{-(\gamma-1)s \log n}$ . Note that this is still true after a union bound over all the possible ways to distributed the  $O(c \log^{3/2} n)$  tokens among the  $\leq s$  nodes. Using Lemma 3, we then have to add at most  $O(s) = O(c \log n)$  additional non-free edges to make the graph induced by  $S$   $c$ -vertex connected.

There are at most  $n^s = 2^{s \log n}$  ways to choose the set  $S$  and  $k^s = 2^{O(s \log n)}$  ways to assign tokens to the nodes in  $S$ . Hence, if we choose  $\gamma$  sufficiently large, the probability that we need to increase the potential by at most  $O(c \log^{3/2} n)$  for all sets  $S$  and all token assignments is positive. The theorem now follows as in the previous lower bounds (e.g., as in the proof of Theorem 1).  $\square$

**Acknowledgments.** We would like to thank Chinmoy Dutta, Gopal Pandurangan, Rajmohan Rajaraman, and Zhifeng Sun for helpful discussions and for sharing their work at an early stage.

## References

1. Anta, A.F., Milani, A., Mosteiro, M.A., Zaks, S.: Opportunistic Information Dissemination in Mobile Ad-hoc Networks: The Profit of Global Synchrony. In: Lynch, N.A., Shvartsman, A.A. (eds.) DISC 2010. LNCS, vol. 6343, pp. 374–388. Springer, Heidelberg (2010)

2. Avin, C., Koucký, M., Lotker, Z.: How to Explore a Fast-Changing World (Cover Time of a Simple Random Walk on Evolving Graphs). In: Aceto, L., Damgård, I., Goldberg, L.A., Halldórsson, M.M., Ingólfssdóttir, A., Walukiewicz, I. (eds.) ICALP 2008, Part I. LNCS, vol. 5125, pp. 121–132. Springer, Heidelberg (2008)
3. Baumann, H., Crescenzi, P., Fraigniaud, P.: Parsimonious flooding in dynamic graphs. In: Proc. 28th ACM Symp. on Principles of Distributed Computing (PODC), pp. 260–269 (2009)
4. Clementi, A., Macci, C., Monti, A., Pasquale, F., Silvestri, R.: Flooding time in edge-markovian dynamic graphs. In: Proc. of 27th ACM Symp. on Principles of Distributed Computing (PODC), pp. 213–222 (2008)
5. Clementi, A., Monti, A., Pasquale, F., Silvestri, R.: Broadcasting in dynamic radio networks. *Journal of Computer and System Sciences* 75(4), 213–230 (2009)
6. Clementi, A., Pasquale, F., Monti, A., Silvestri, R.: Information spreading in stationary markovian evolving graphs. In: Proc. of IEEE Symp. on Parallel & Distributed Processing, IPDPS (2009)
7. Clementi, A., Silvestri, R., Trevisan, L.: Information spreading in dynamic graphs. In: Proc. 31st Symp. on Principles of Distributed Computing, PODC (2012)
8. Cornejo, A., Gilbert, S., Newport, C.: Aggregation in dynamic networks. In: Proc. 31st Symp. on Principles of Distributed Computing, PODC (2012)
9. Dutta, C., Pandurangan, G., Rajaraman, R., Sun, Z.: Information spreading in dynamic networks. CoRR, abs/1112.0384 (2011)
10. Haeupler, B.: Analyzing network coding gossip made easy. In: Proc. 43rd Symp. on Theory of Computing (STOC), pp. 293–302 (2011)
11. Haeupler, B., Karger, D.: Faster information dissemination in dynamic networks via network coding. In: Proc. 30th Symp. on Principles of Distributed Computing (PODC), pp. 381–390 (2011)
12. Haeupler, B., Médard, M.: One packet suffices - highly efficient packetized network coding with finite memory. In: 2011 IEEE International Symposium on Information Theory Proceedings (ISIT), pp. 1151–1155 (2011)
13. Jackson, B., Jordán, T.: Independence free graphs and vertex connectivity augmentation. *Journal of Combinatorial Theory, Series B* 94(1), 31–77 (2005)
14. Kuhn, F., Lenzen, C., Locher, T., Oshman, R.: Optimal gradient clock synchronization in dynamic networks. In: Proc. of 29th ACM Symp. on Principles of Distributed Computing (PODC), pp. 430–439 (2010)
15. Kuhn, F., Lynch, N., Newport, C., Oshman, R., Richa, A.: Broadcasting in unreliable radio networks. In: Proc. of 29th ACM Symp. on Principles of Distributed Computing (PODC), pp. 336–345 (2010)
16. Kuhn, F., Lynch, N., Oshman, R.: Distributed computation in dynamic networks. In: Proc. 42nd Symp. on Theory of Computing (STOC), pp. 557–570 (2010)
17. Kuhn, F., Moses, Y., Oshman, R.: Coordinated consensus in dynamic networks. In: Proc. 30th Symp. on Principles of Distributed Computing (PODC), pp. 1–10 (2011)
18. Kuhn, F., Oshman, R.: Dynamic networks: Models and algorithms. *SIGACT News* 42(1), 82–96 (2011)
19. O’Dell, R., Wattenhofer, R.: Information dissemination in highly dynamic graphs. In: Proc. of Workshop on Foundations of Mobile Computing (DIALM-POMC), pp. 104–110 (2005)