Chapter 6
Consensus

Distributed Systems

SS 2015

Fabian Kuhn

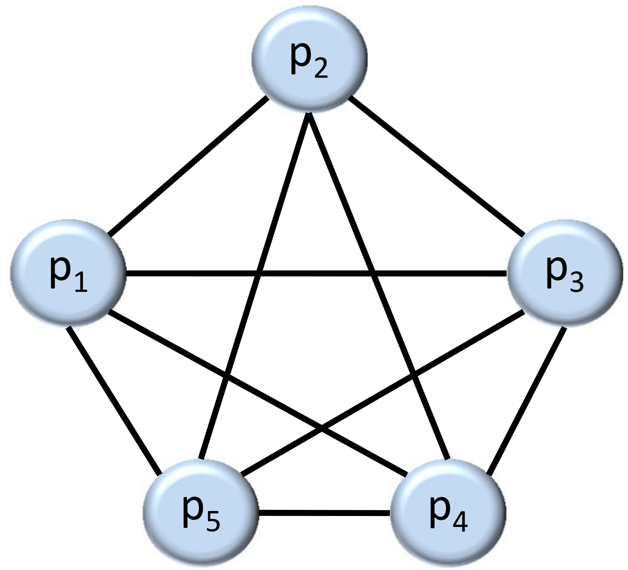
Overview

- Introduction
- Consensus #1: Shared Memory
- Consensus #2: Wait-free Shared Memory
- Consensus #3: Read-Modify-Write Shared Memory
- Consensus #4: Synchronous Systems
- Consensus #5: Byzantine Failures
- Consensus #6: A Simple Algorithm for Byzantine Agreement
- Consensus #7: The Queen Algorithm
- Consensus #8: The King Algorithm
- Consensus #9: Byzantine Agreement Using Authentication
- Consensus #10: A Randomized Algorithm
- Shared Coin

- Slides by R. Wattenhofer (ETHZ)

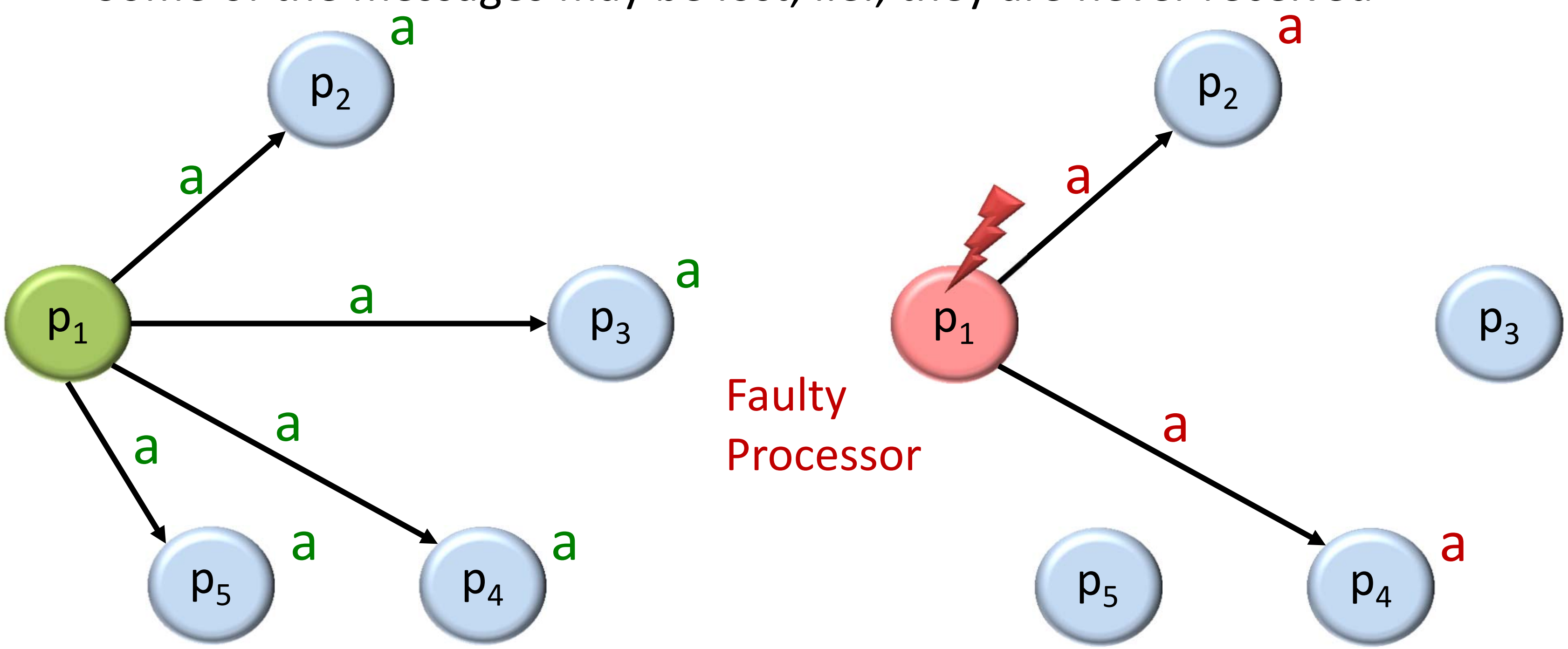
Consensus #4: Synchronous Systems

- One can sometimes tell if a processor had crashed
  - Timeouts
  - Broken TCP connections
- Can one solve consensus at least in synchronous systems?
- Model
  - All communication occurs in synchronous rounds
  - Complete communication graph

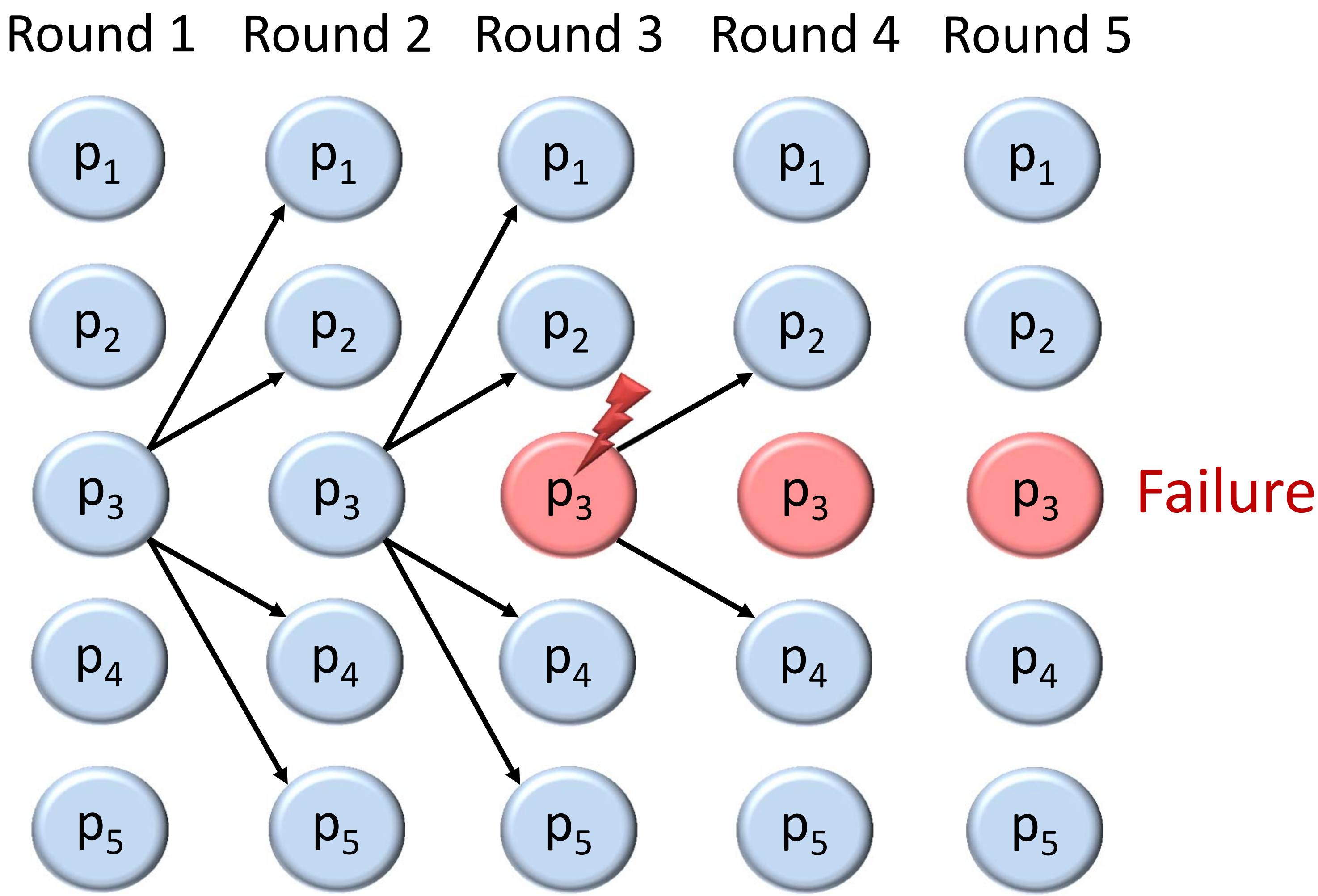


Crash Failures

- Broadcast: Send a message to all nodes in one round
  - At the end of the round everybody receives the message a
  - Every process can broadcast a value in each round
- Crash Failures: A broadcast can fail if a process crashes
  - Some of the messages may be lost, i.e., they are never received

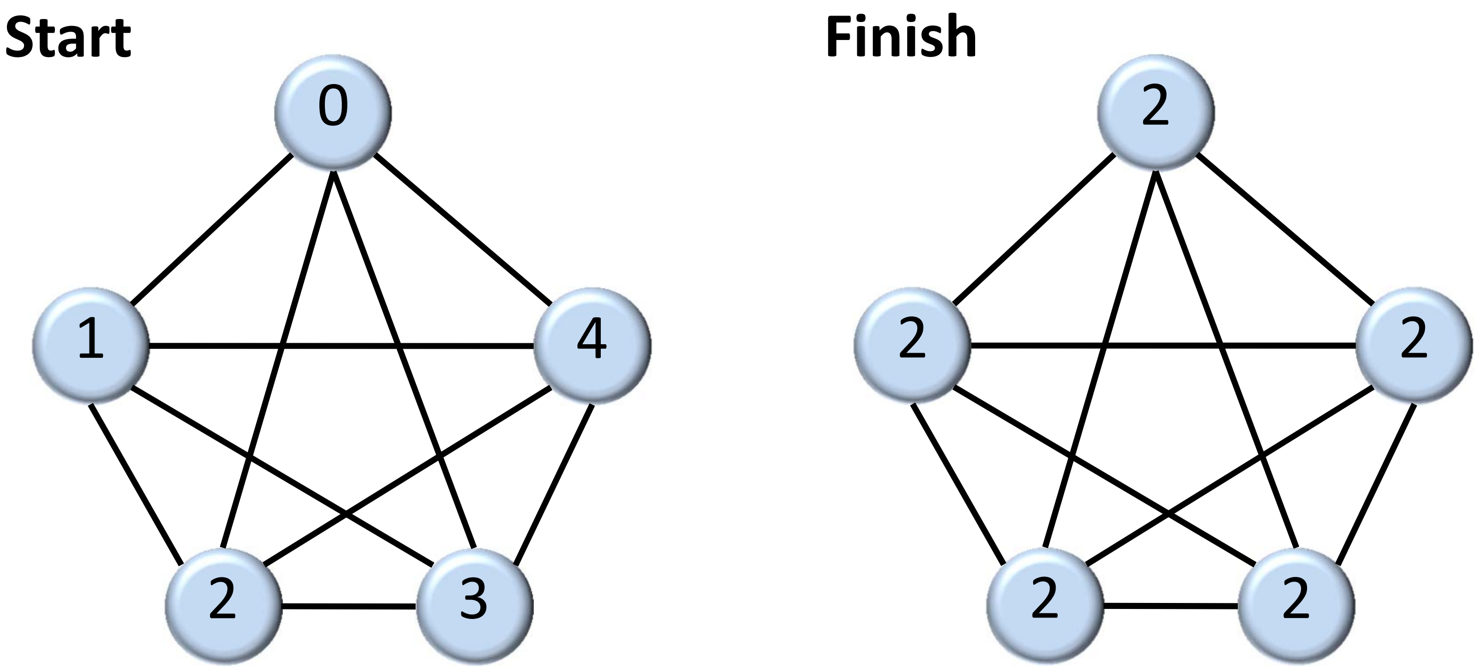


Process disappears after failure



Consensus Repetition

- **Input:** everybody has an initial value
- **Agreement:** everybody must decide on the same value



- **Validity conditon**: If everybody starts with the same value, everybody must decide on that value

A Simple Consensus Algorithm

Each process:

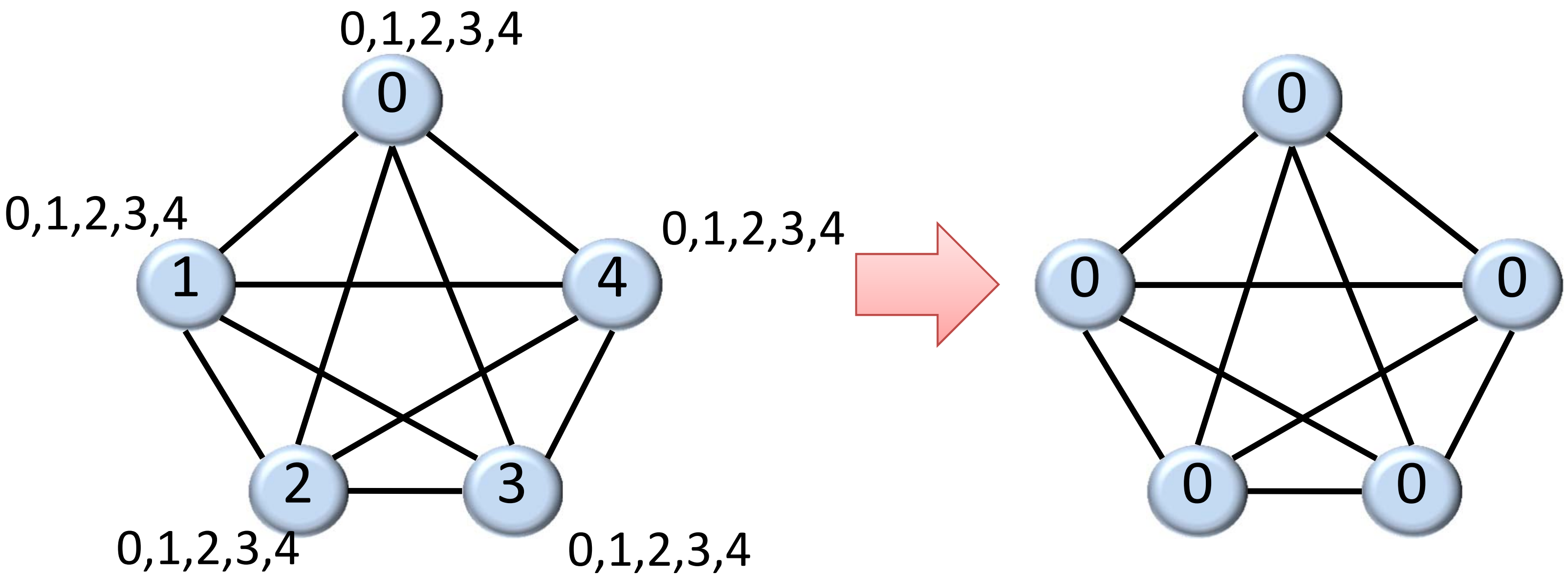
1. Broadcast own value
2. Decide on the minimum of all received values

Including the own value

Note that only one round is needed!

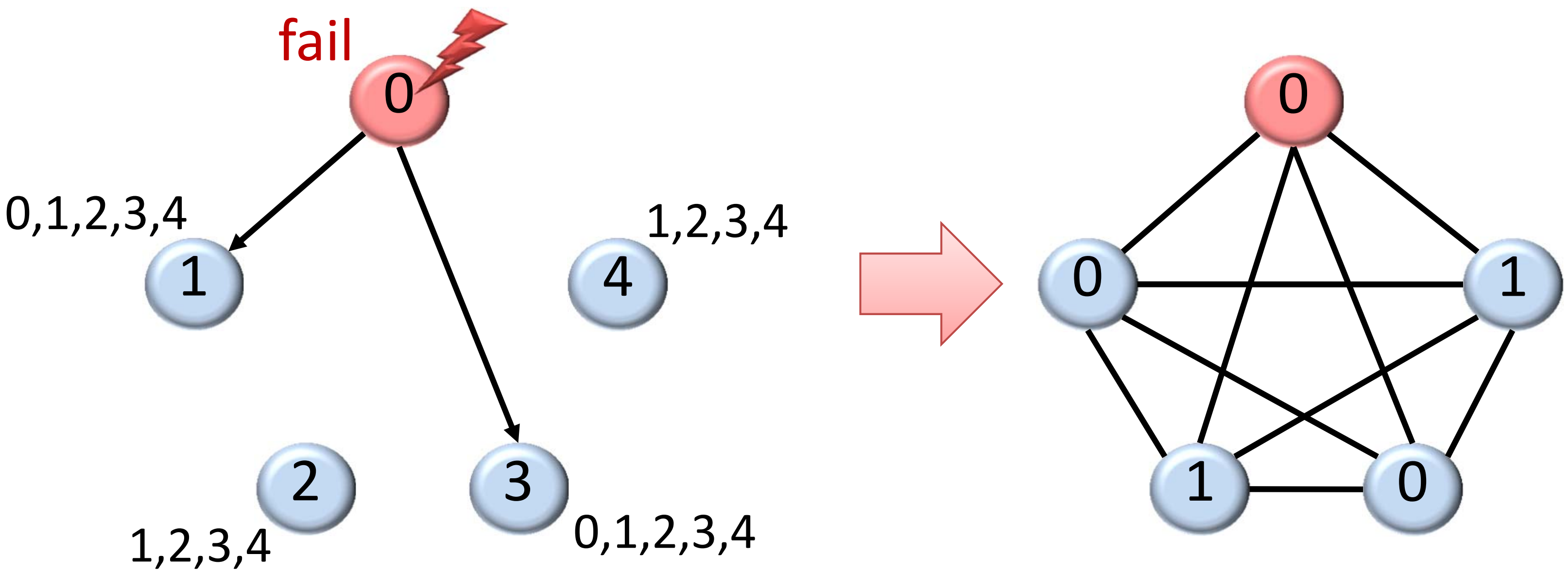
Execution Without Failures

- Broadcast values and decide on minimum → Consensus!
- Validity condition is satisfied: If everybody starts with the same initial value, everybody sticks to that value (minimum)



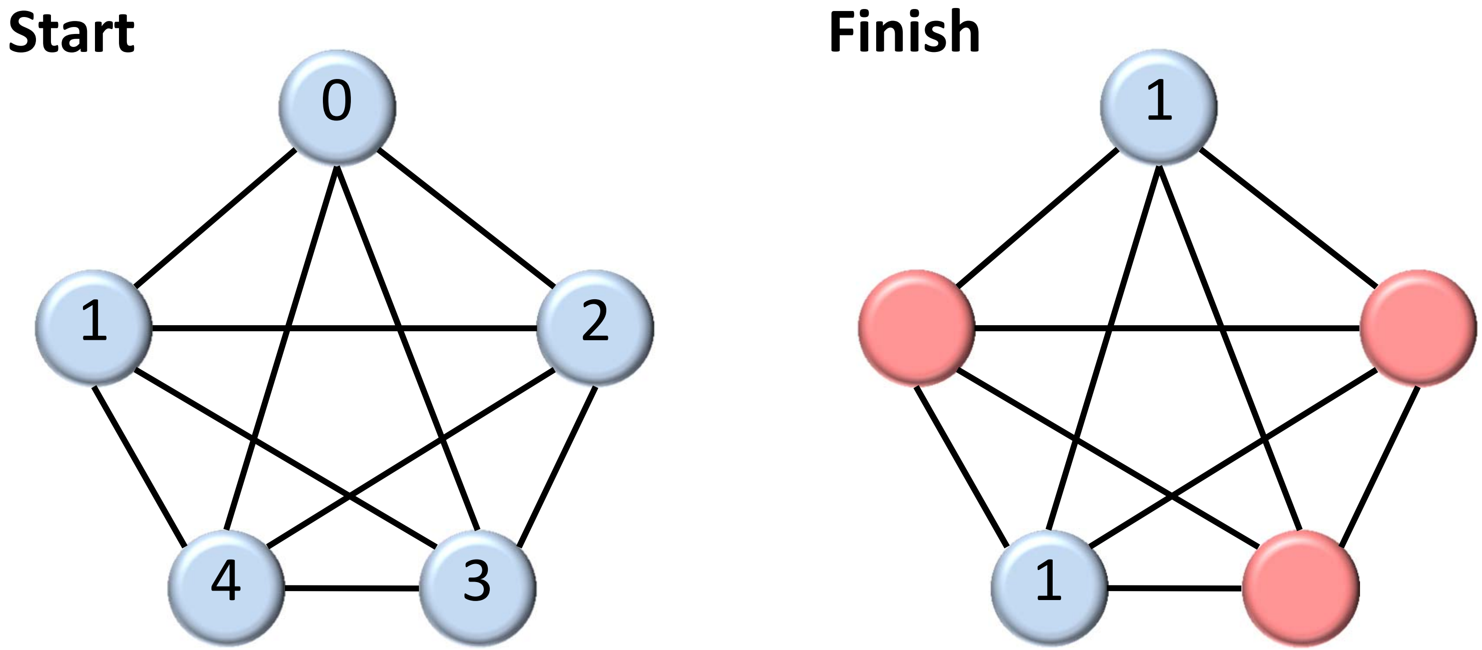
Execution With Failures

- The failed processor doesn't broadcast its value to all processors
- Decide on minimum → No consensus!



f-Resilient Consensus Algorithm

- If an algorithm solves consensus for f failed processes, we say it is an f-resilient consensus algorithm
- Example: The input and output of a 3-resilient consensus alg.



- **Refined validity condition**:
  All processes decide on a value that is available initially

An f-Resilient Consensus Algorithm

Each process:

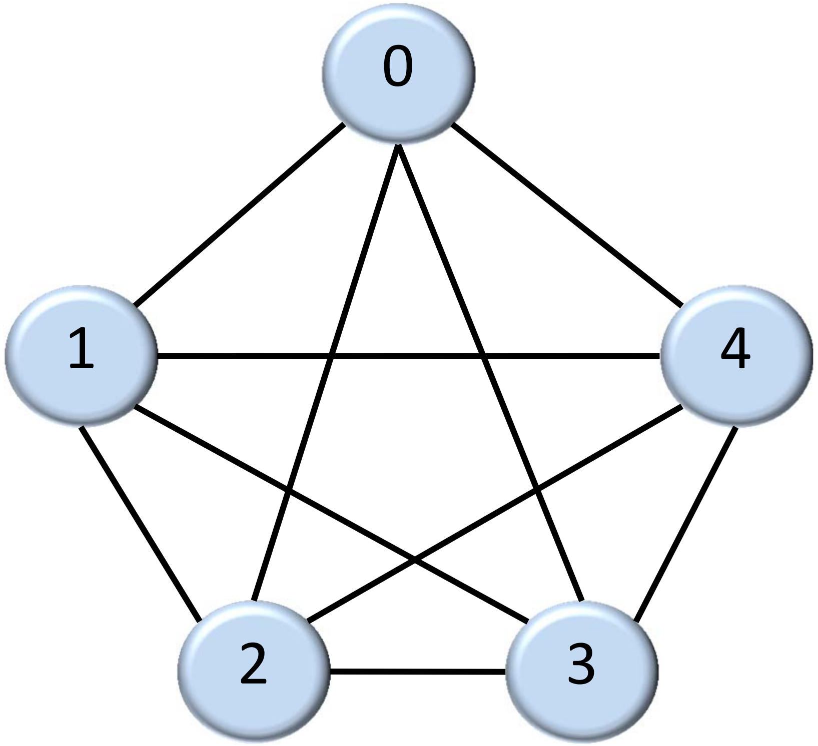
Round 1:
Broadcast own value

Round 2 to round $f+1$:
Broadcast the minimum of the received values unless it has been sent before

End of round $f+1$:
Decide on the minimum value received

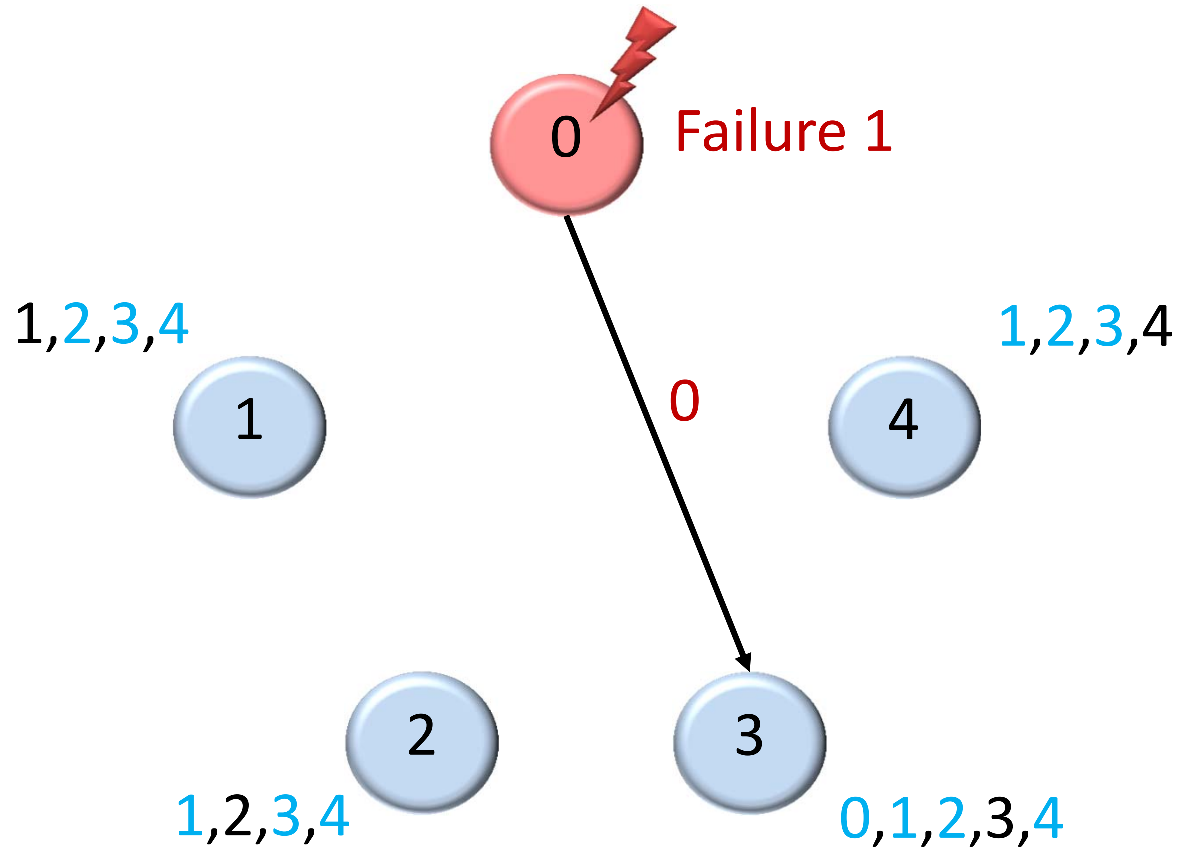
An f-Resilient Consensus Algorithm

- Example: $f = 2$ failures, $f + 1 = 3$ rounds needed



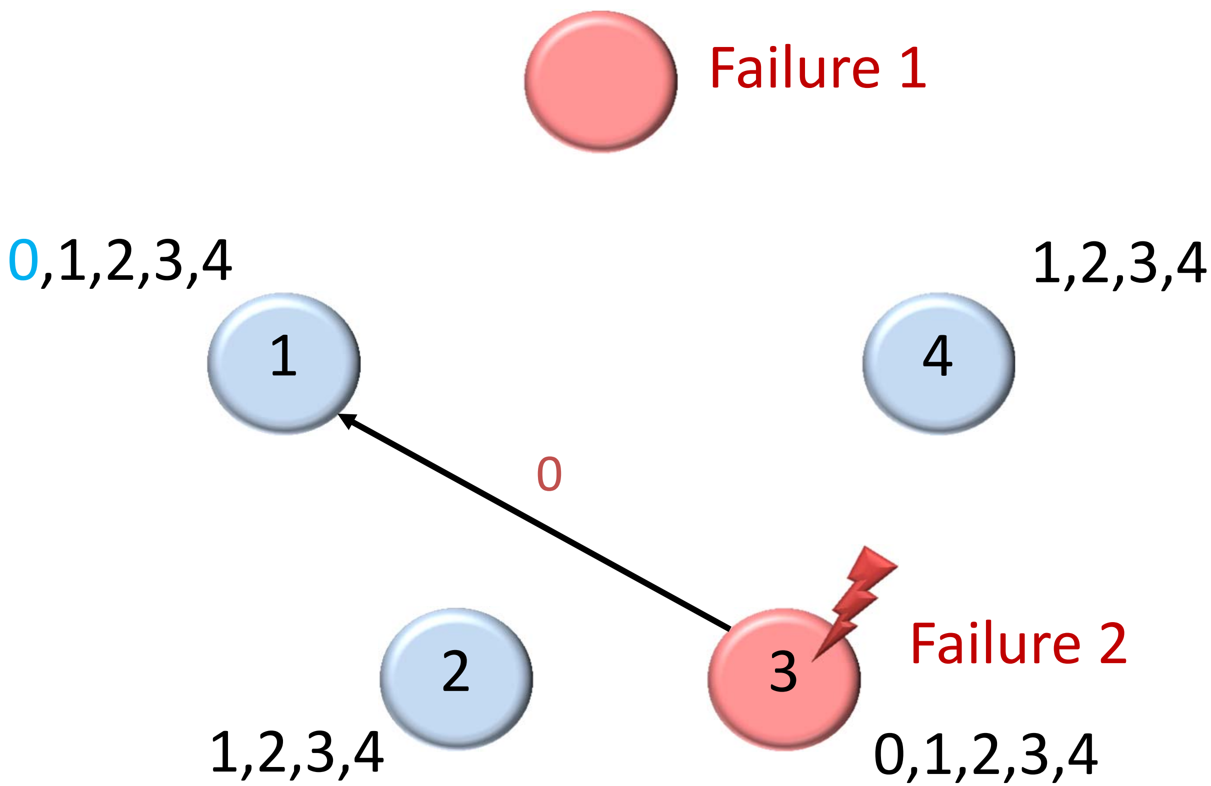
An f-Resilient Consensus Algorithm

- Round 1: Broadcast all values to everybody



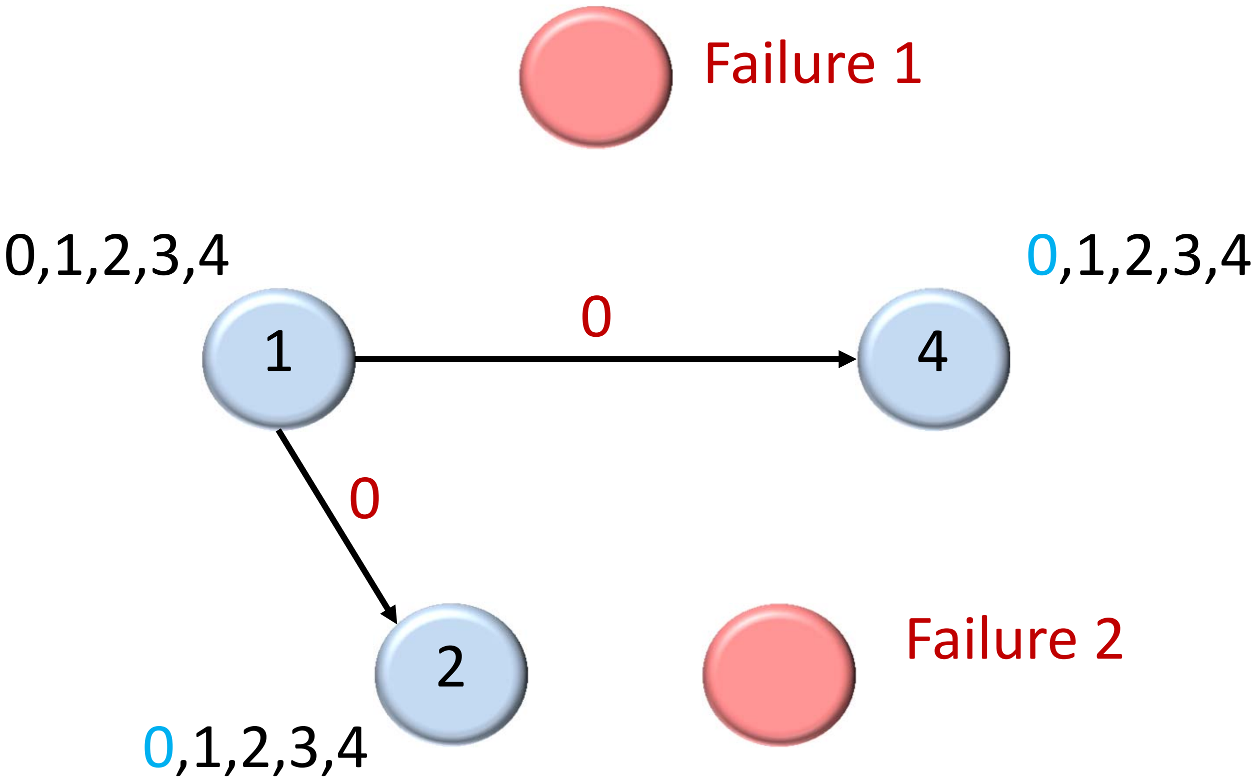
An f-Resilient Consensus Algorithm

- Round 2: Broadcast all new values to everybody



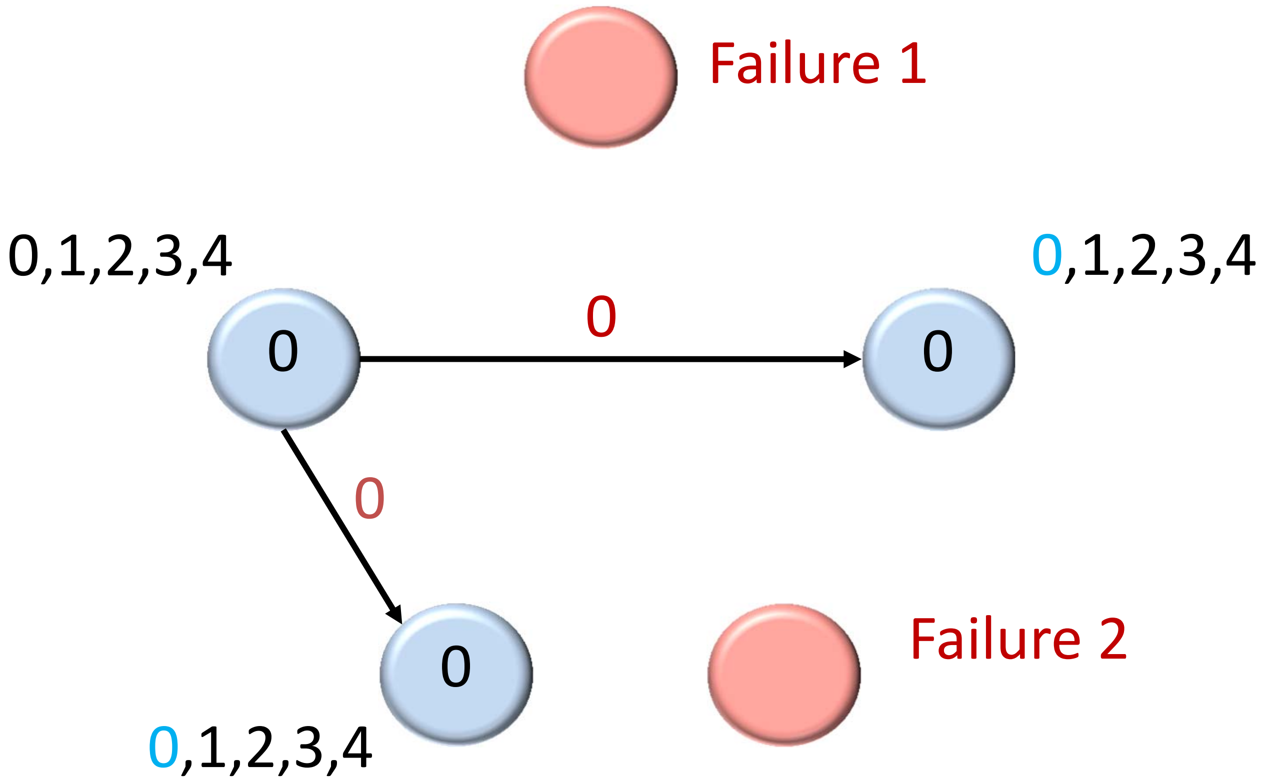
An f-Resilient Consensus Algorithm

- Round 3: Broadcast all new values to everybody



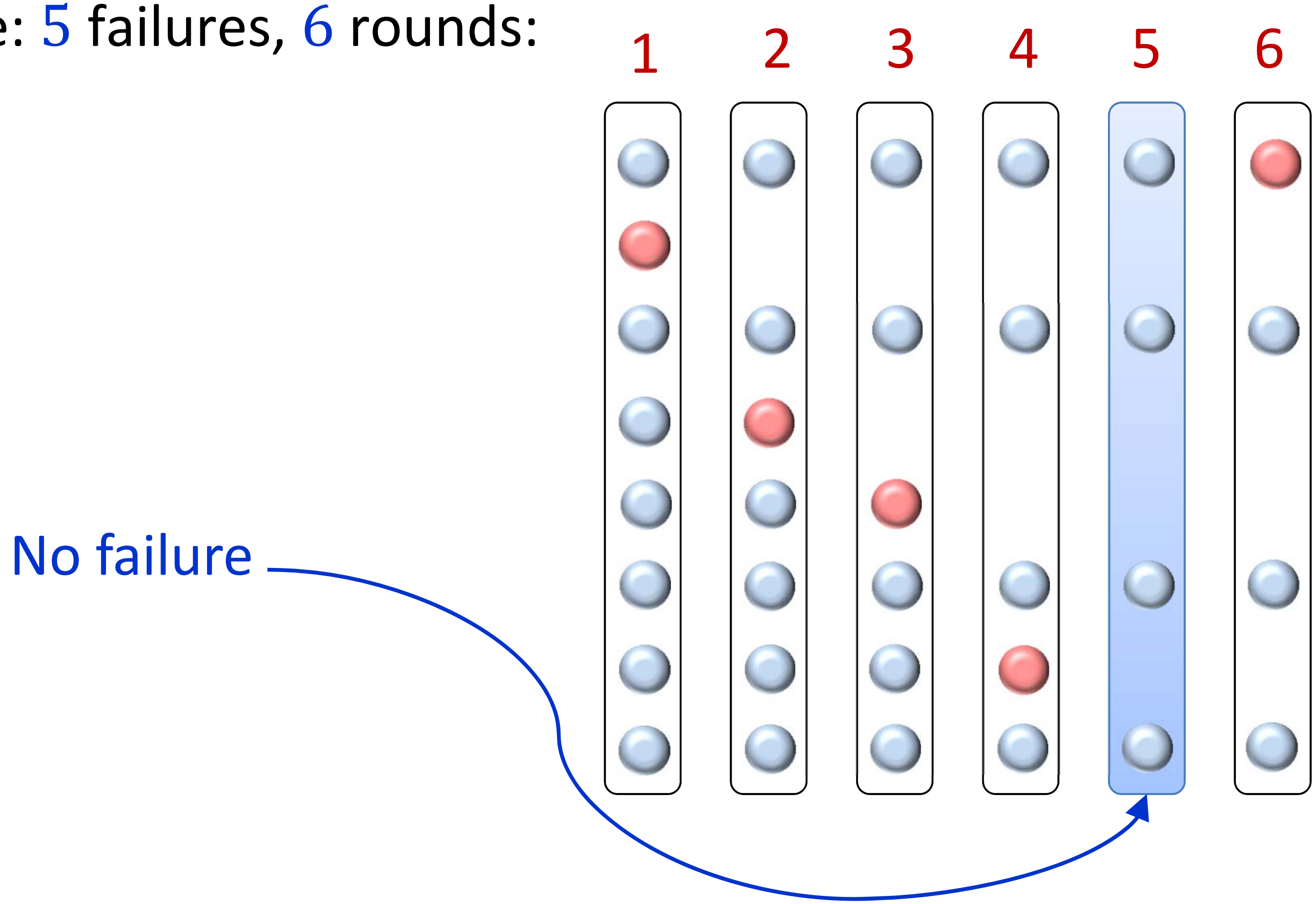
An f-Resilient Consensus Algorithm

- Decide on minimum → Consensus!



Analysis

- If there are f failures and $f + 1$ rounds, then there is a round with no failed process
- Example: 5 failures, 6 rounds:



Analysis

- At the end of the round with no failure
  - Every (non faulty) process knows about all the values of all the other participating processes
  - This knowledge doesn't change until the end of the algorithm
- Therefore, everybody will decide on the same value
- However, as we don't know the exact position of this round, we have to let the algorithm execute for $f + 1$ rounds

- **Validity:** When all processes start with the same input value, then consensus is that value

Theorem

Theorem

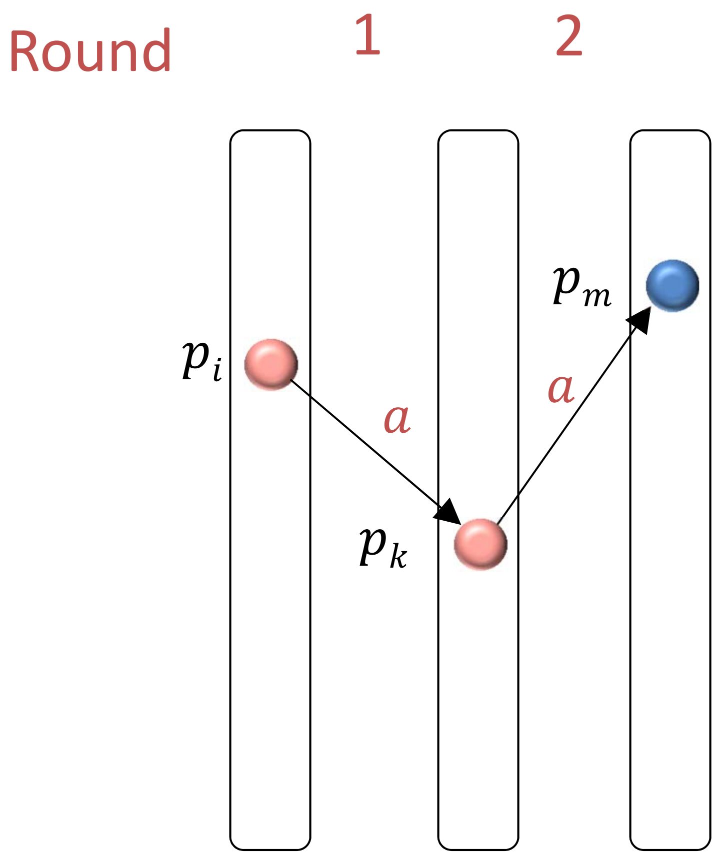
If at most $f \leq n-2$ of n nodes of a synchronous message passing system can crash, at least $f+1$ rounds are needed to solve consensus.

Proof idea:

- Show that f rounds are not enough if $n \geq f+2$
- Before proving the theorem, we consider a

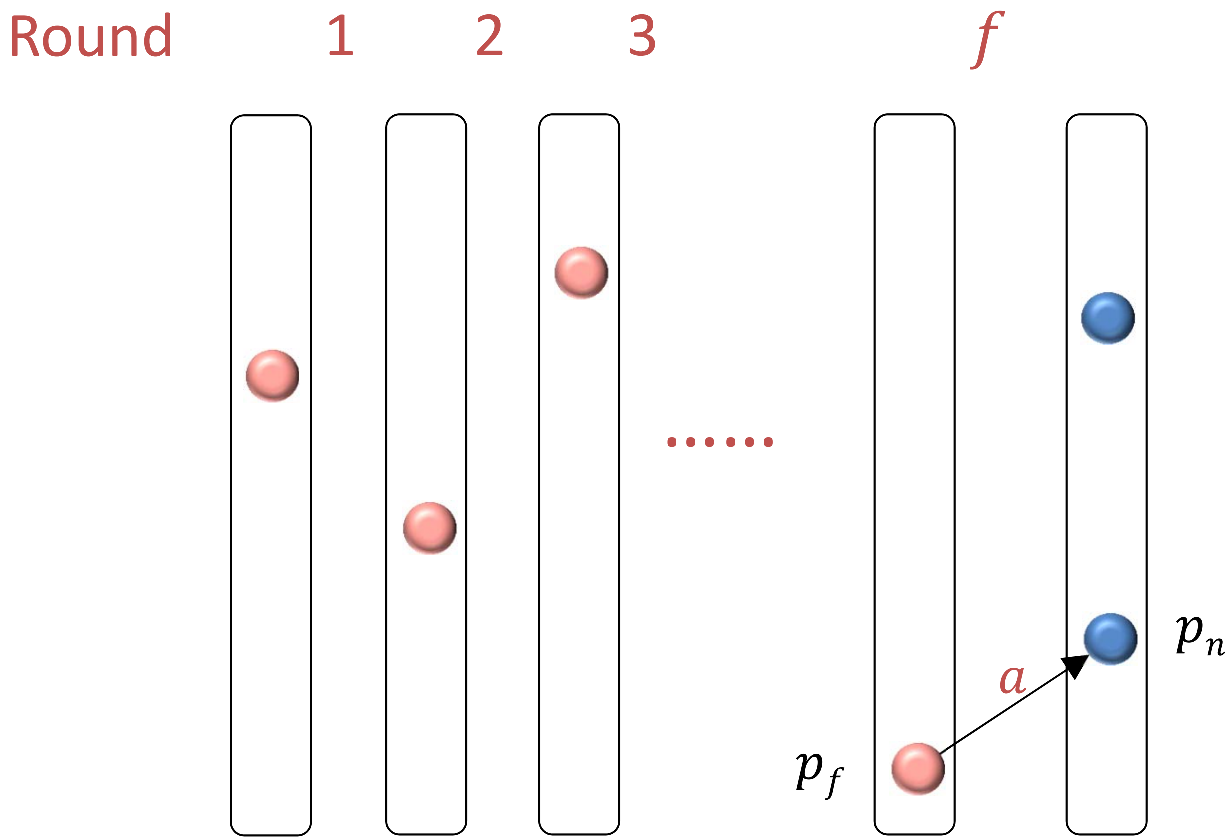
  “worst-case scenario”: In each round one of the processes fails

Lower Bound on Rounds: Intuition



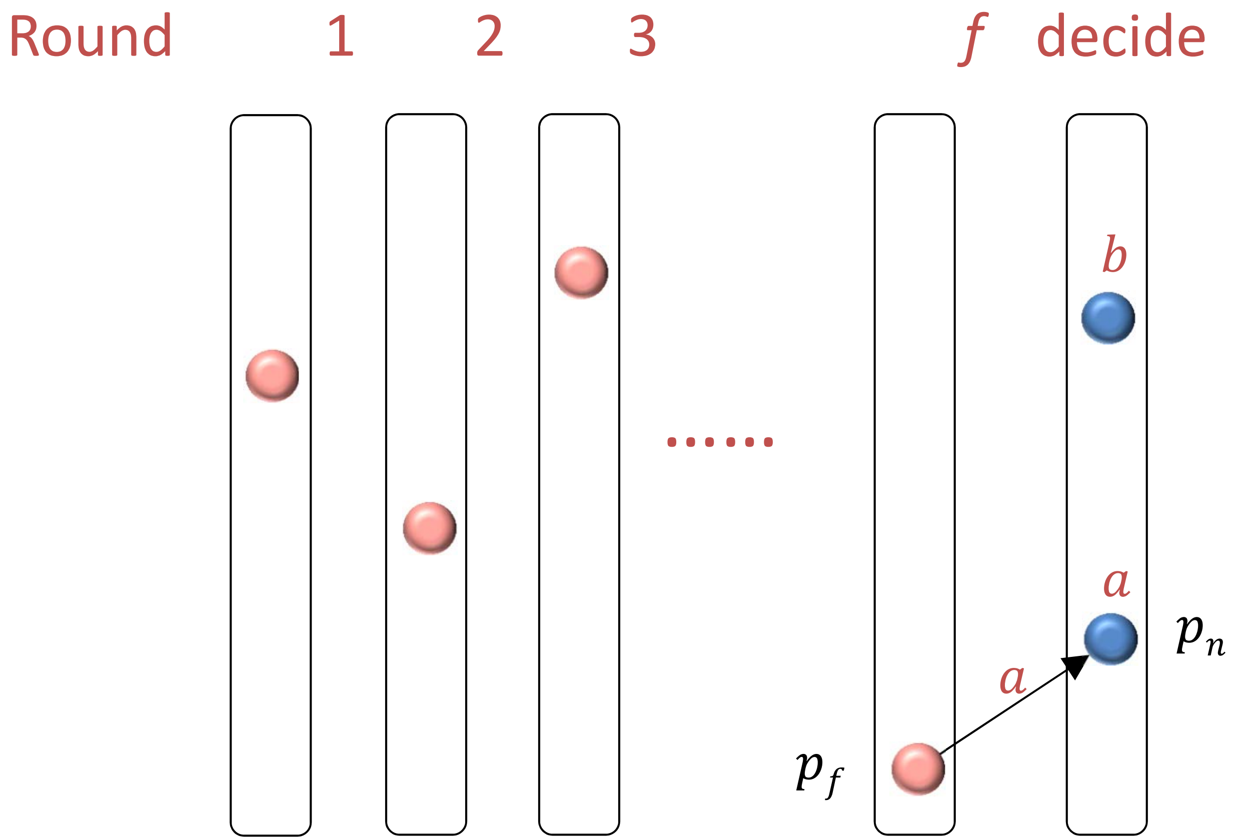
- Before process p_i fails, it sends its value a only to one process p_k
- Before process p_k fails, it sends its value a to only one process p_m

Lower Bound on Rounds: Intuition



- At the end of round f only one process p_n knows about value a

Lower Bound on Rounds: Intuition



- Process p_n may decide on a and all other processes may decide on another value b
- f rounds are not enough
  $\Longrightarrow$ at least $f + 1$ rounds are needed

Lower Bound on Rounds: Proof

Recall (from Chapters 1 & 2):

- For the impossibility proof of the two generals problem, we used an indistinguishability proof
- Execution E is indistinguishable from execution E' for some node v if v sees the same things in both executions.
  - same inputs and messages (schedule)
- If E is indistinguishable from E' for v, then v does the same thing in both executions.
  - We denoted this by $E|v = E'|v$

Similarity:

- Call E_i and E_j **similar** if $E_i|v = E_j|v$ for some node v

$$E_i \sim_v E_j \iff E_i|v = E_j|v$$

Lower Bound on Rounds: Proof

Similarity Chain:

- Consider a sequence of executions $E_1, E_2, E_3, \dots, E_T$ such that

$$\forall \boldsymbol{i} \geq \mathbf{1} \;:\quad \boldsymbol{E_i} \sim_{\boldsymbol{v_i}} \boldsymbol{E_{i+1}}$$

  – any two consecutive executions E_i and E_{i+1} are indistinguishable for some node v_i (we assume that v_i does not crash in E_i and E_{i+1})

- **Indistinguishability:**
  $\forall i \geq 1 \;:\;$ Node v_i decides on the same value in E_i and E_{i+1}

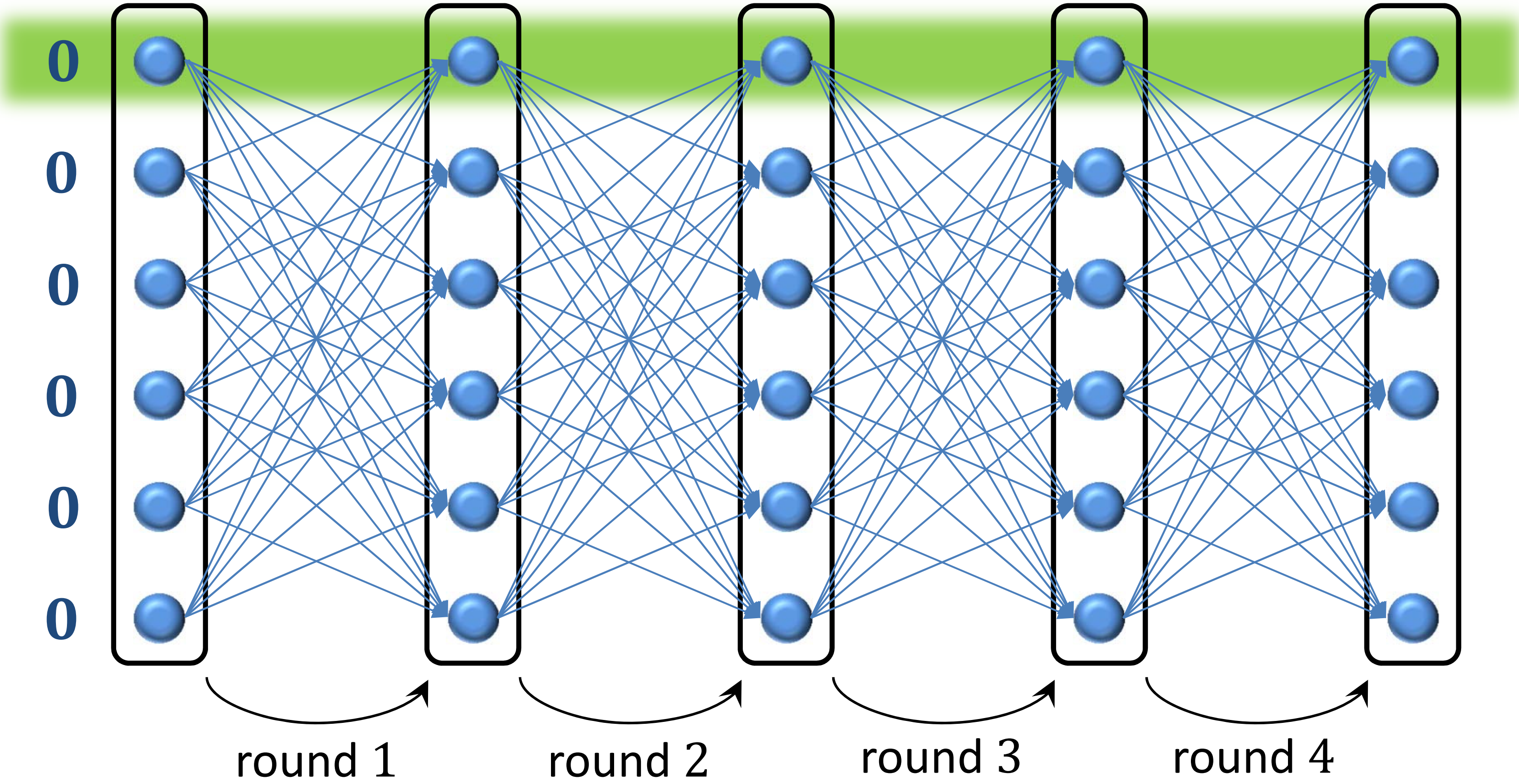
- **Agreement:**
  $\forall i \geq 1 \;:\;$ All nodes decide on the same value in E_i and E_{i+1}

- Hence, all executions $E_1, \dots, E_T$ have the same decision value!

- **Goal:**
  E_1: no crashes, all inputs are 0; E_T: no crashes, all inputs are 1

Lower Bound on Rounds: Proof

Example: $f = 4, n = 6$ **Need to show: 4 rounds are not enough**



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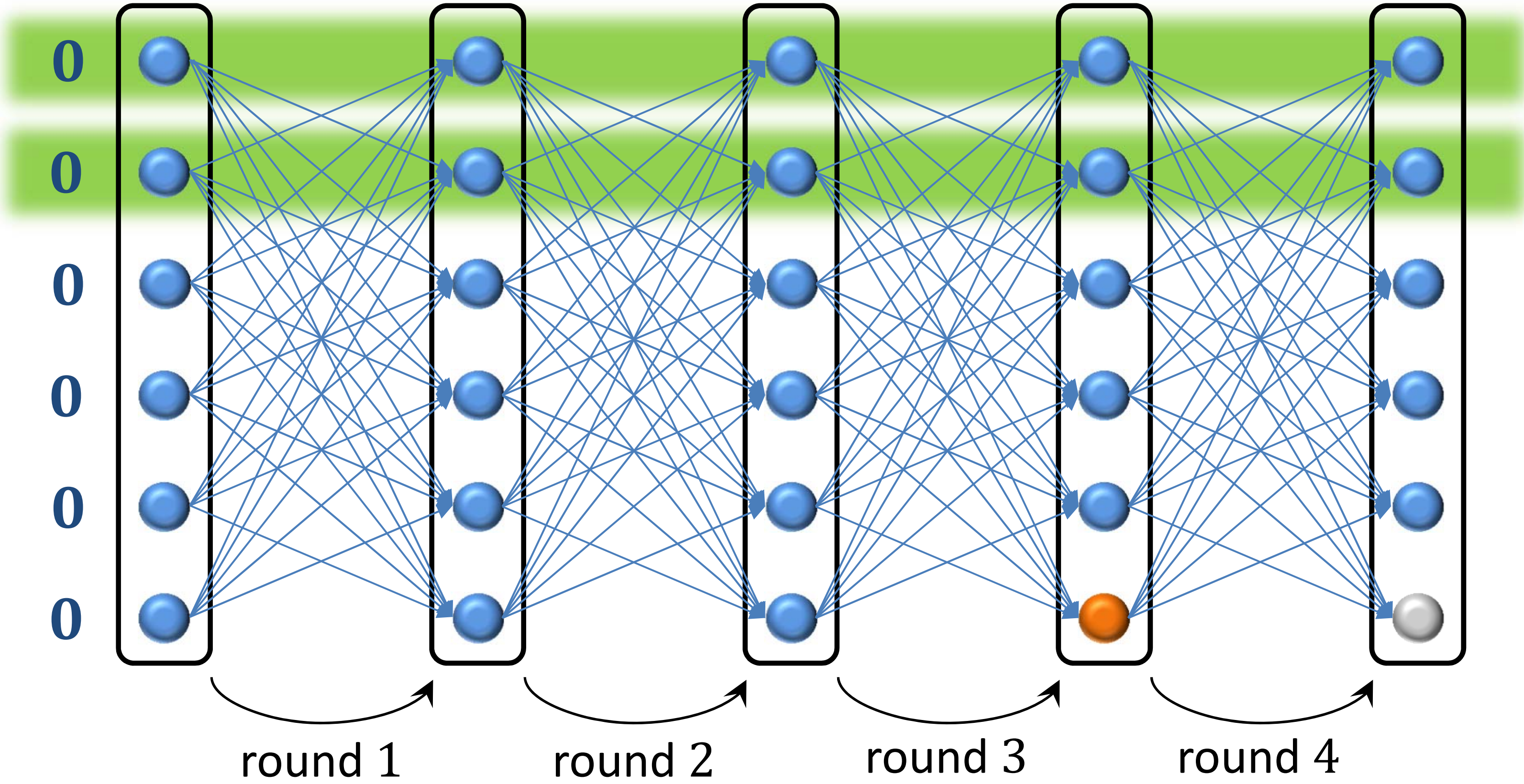

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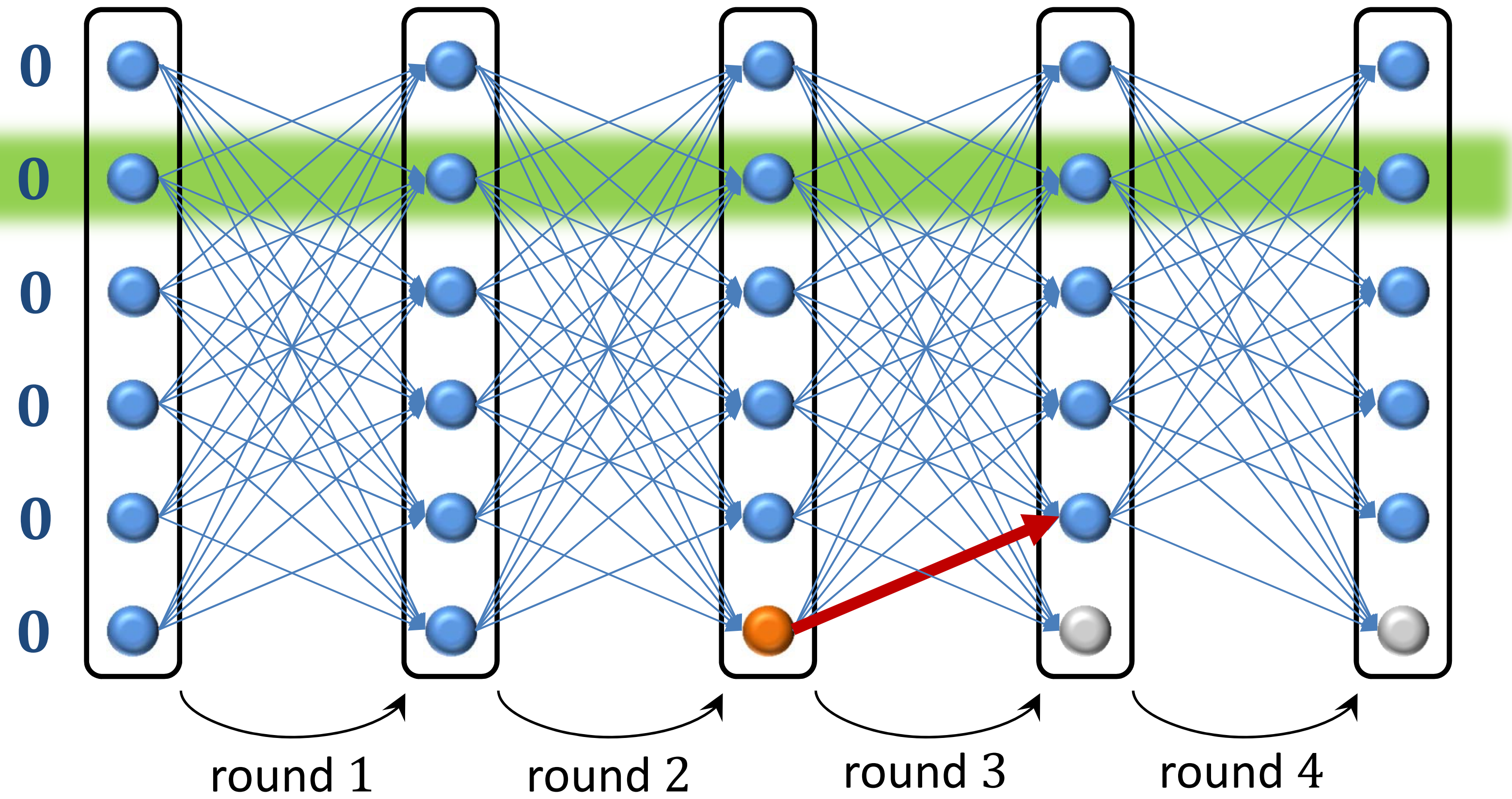

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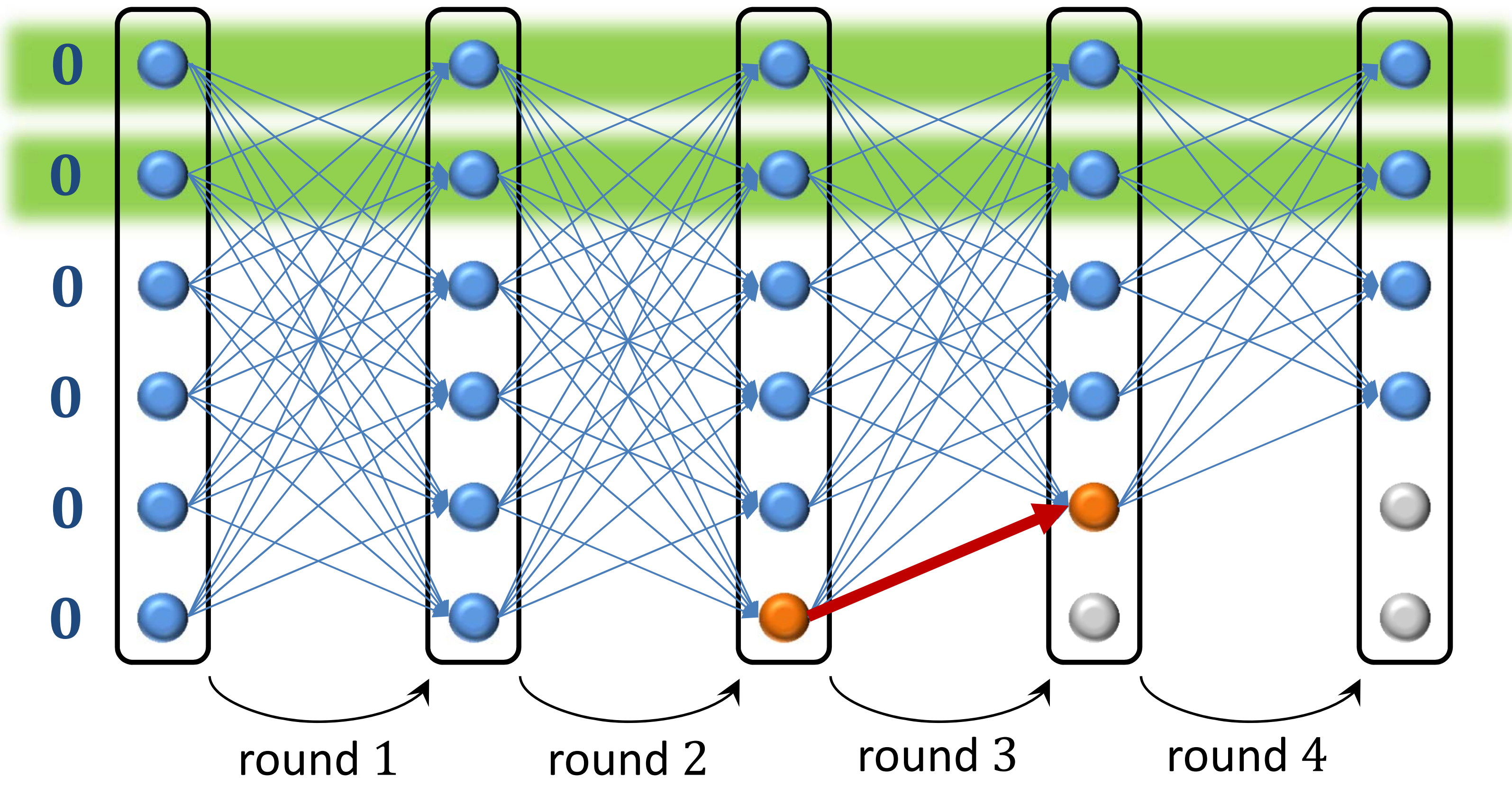

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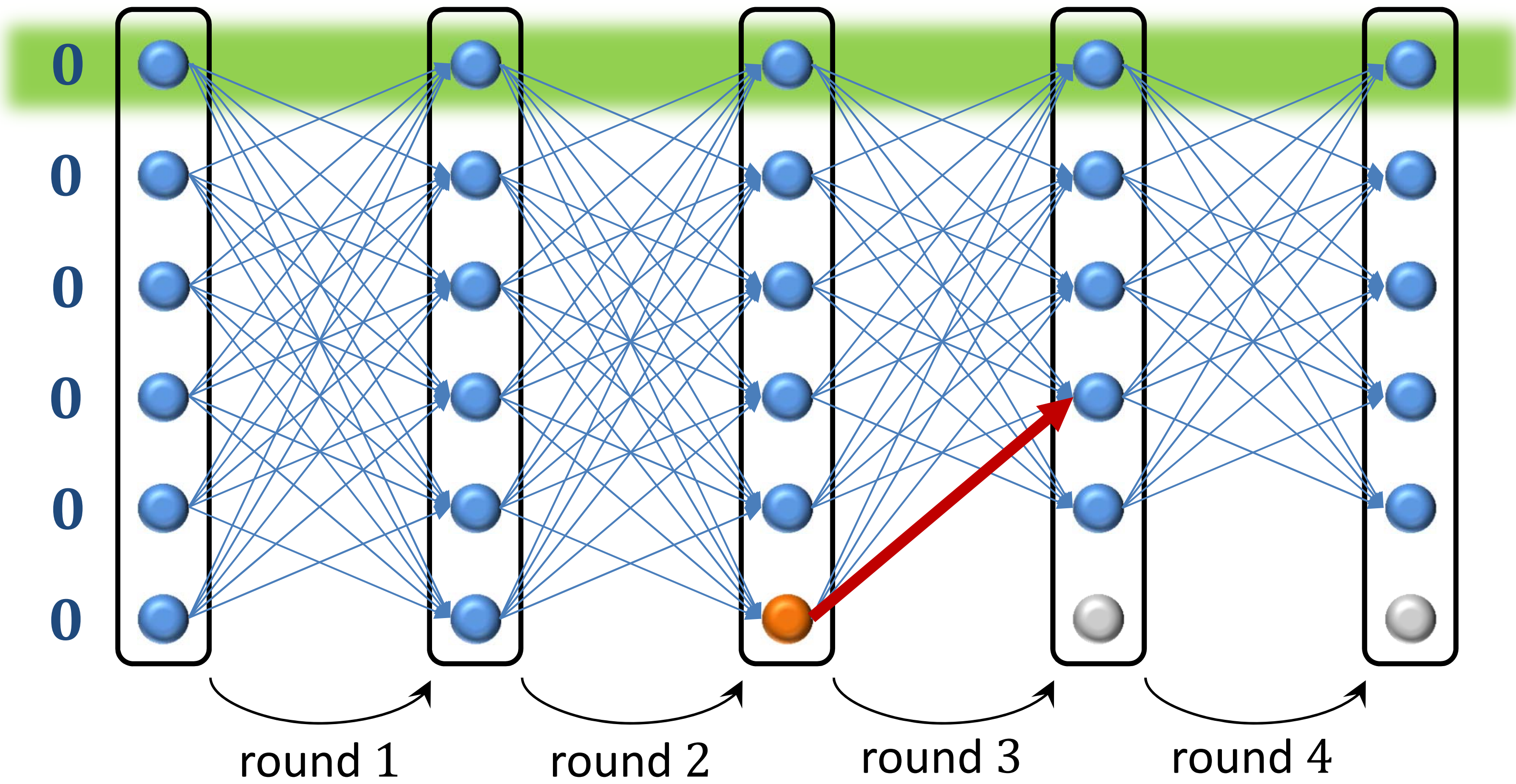

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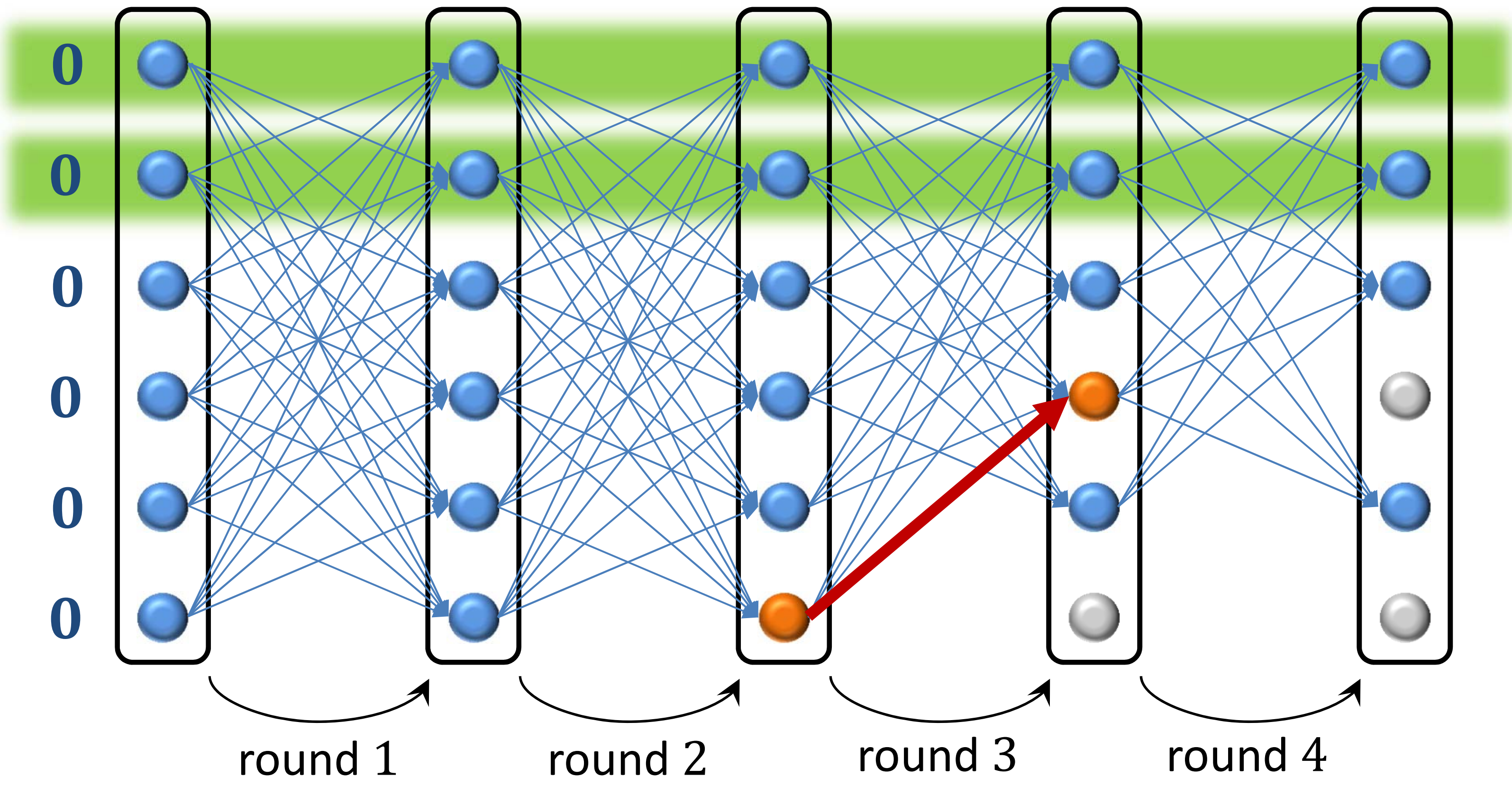

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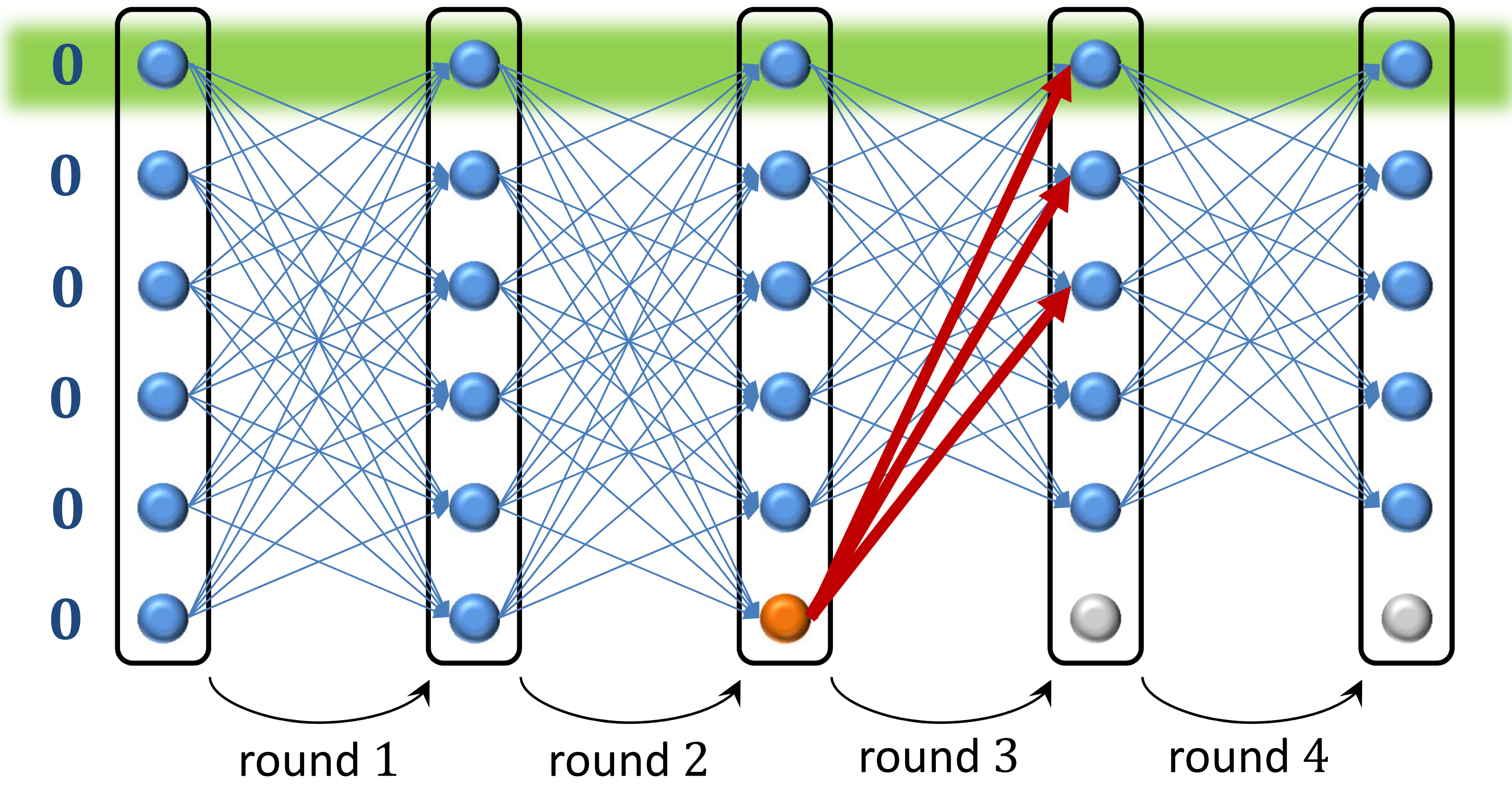

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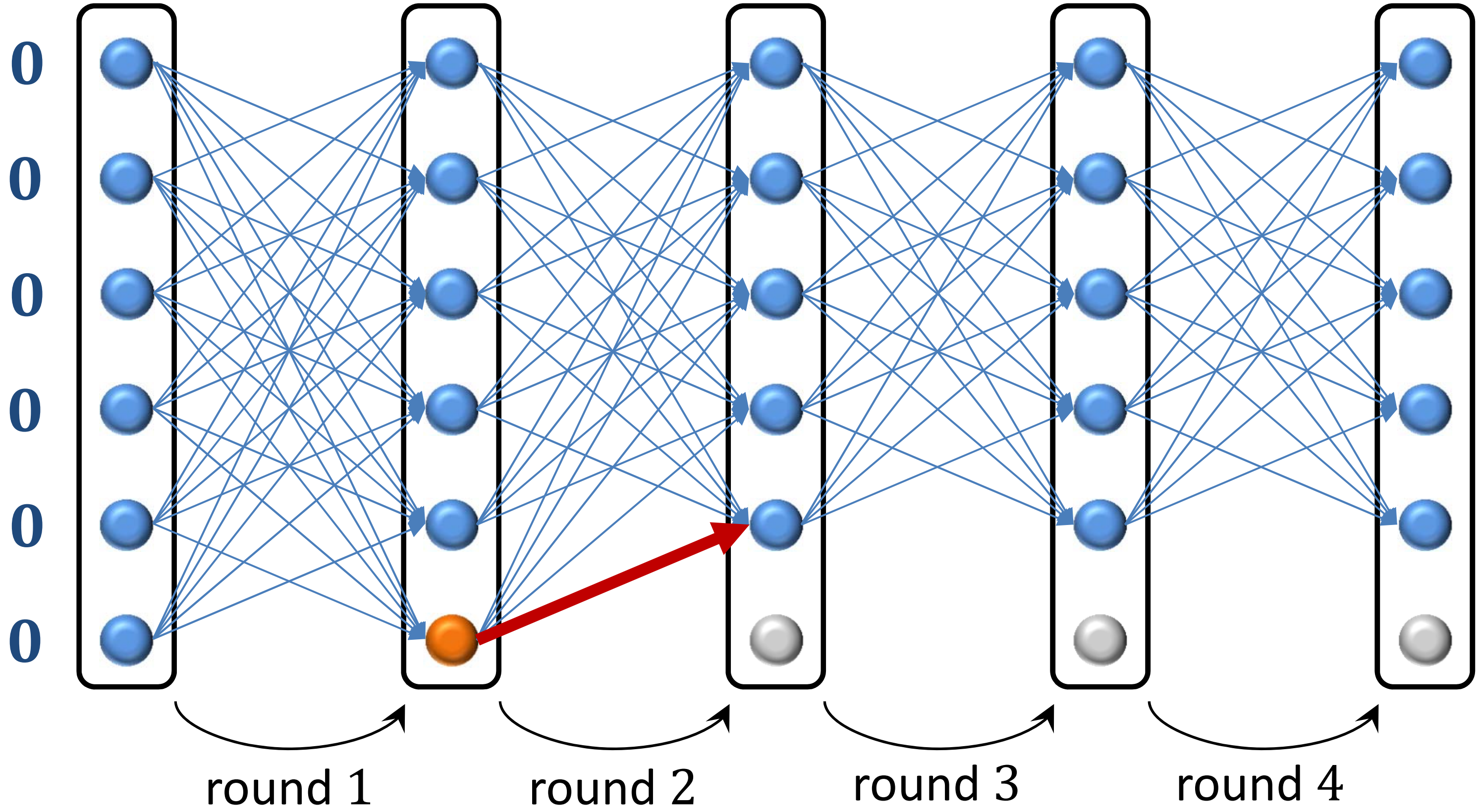

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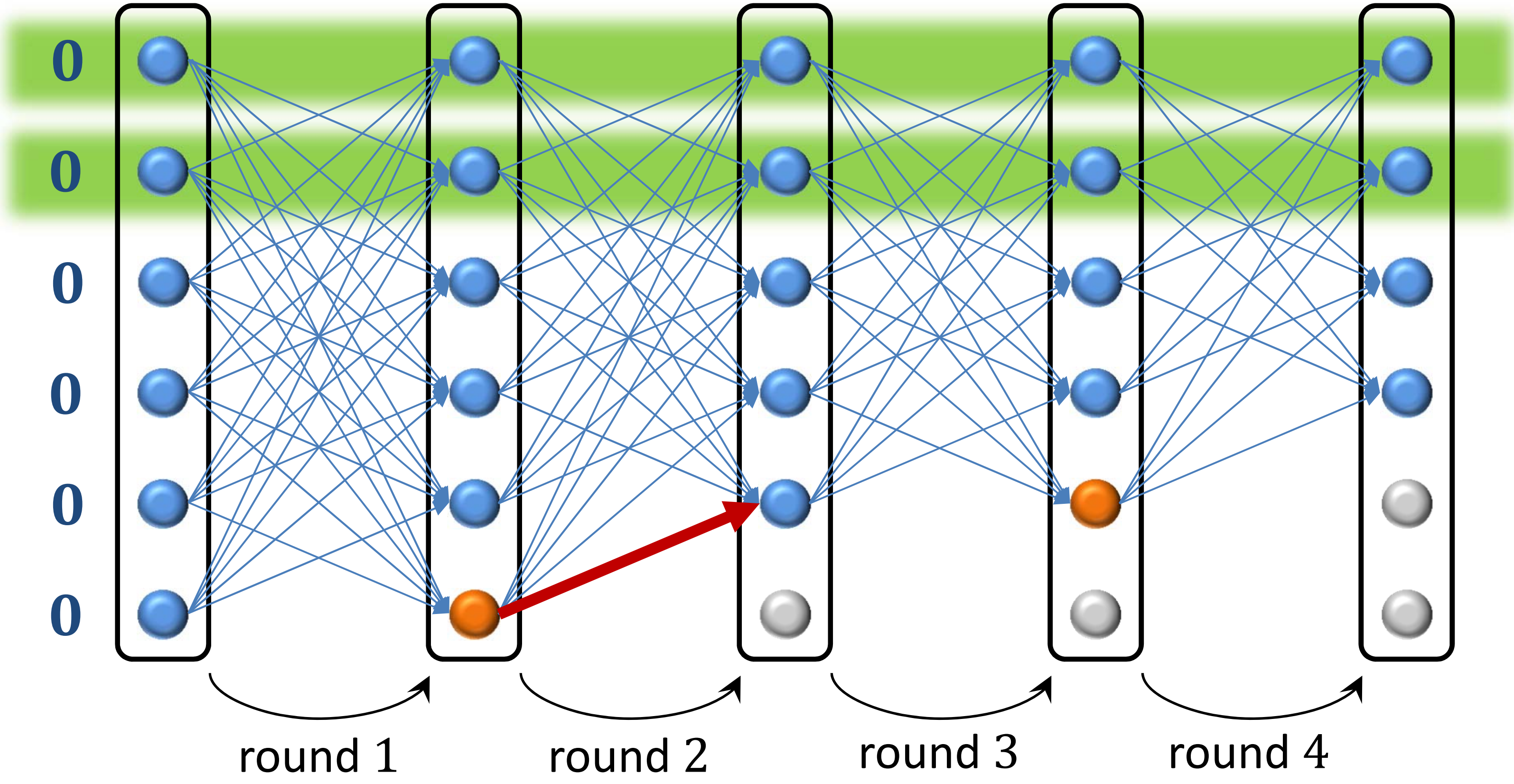

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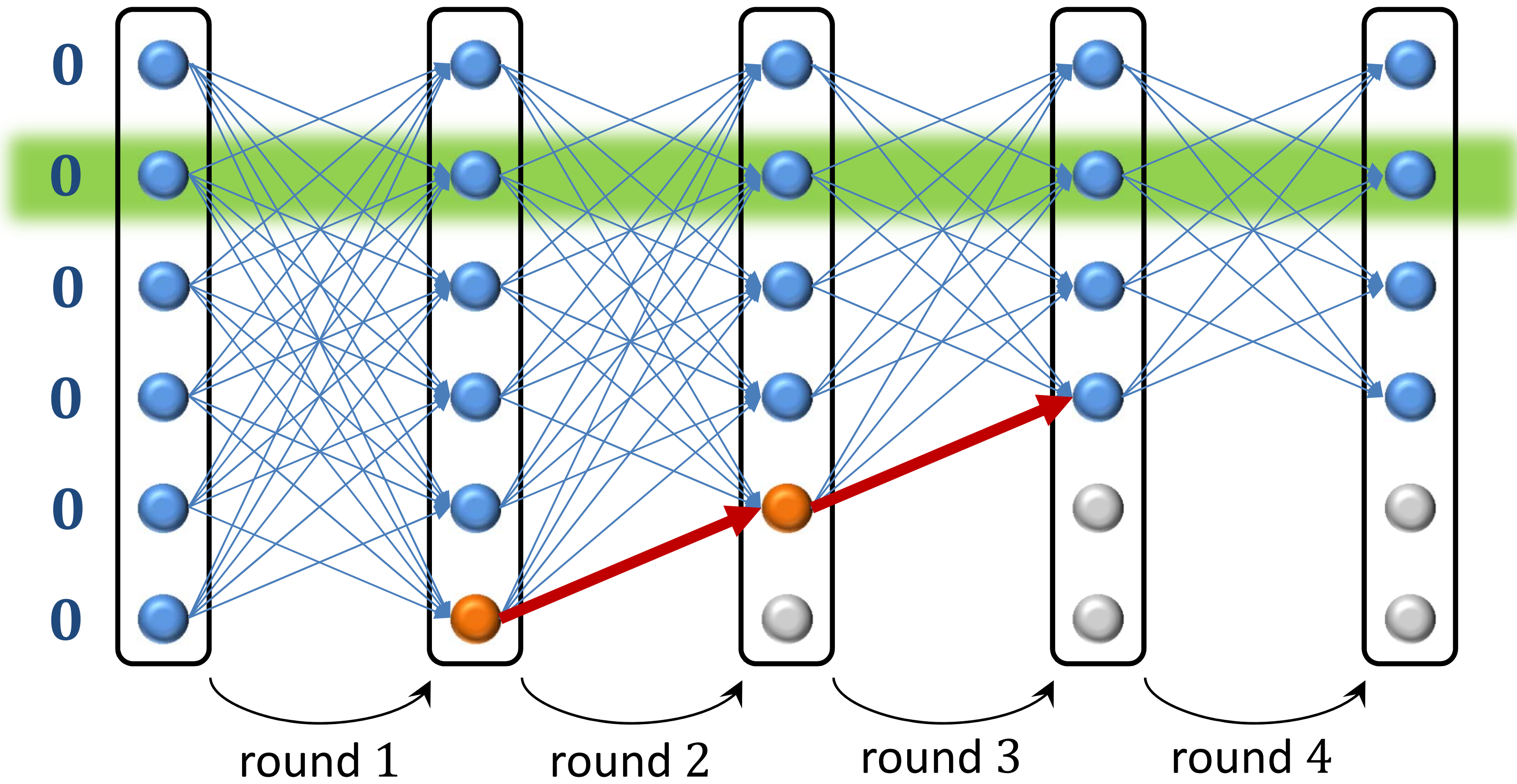

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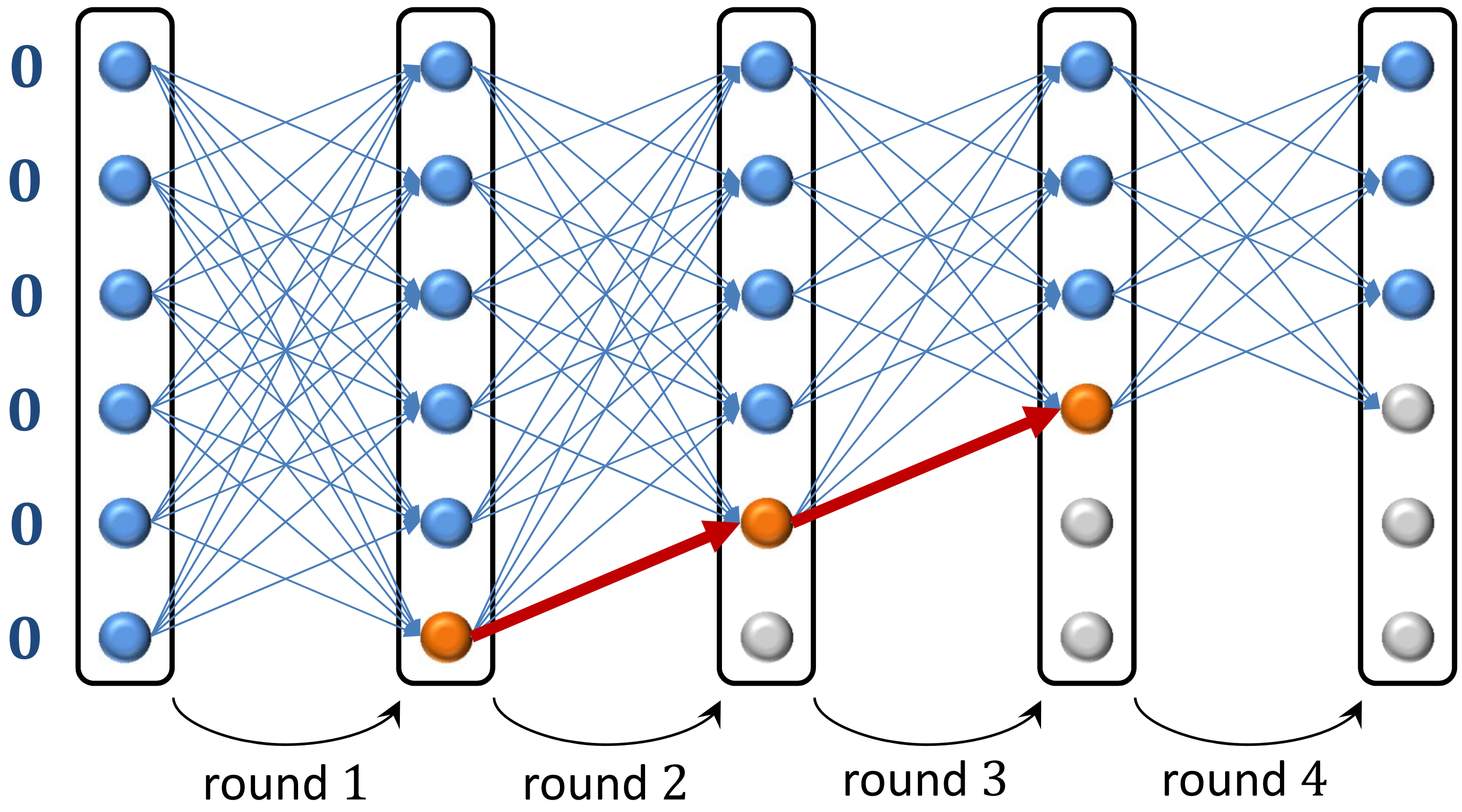

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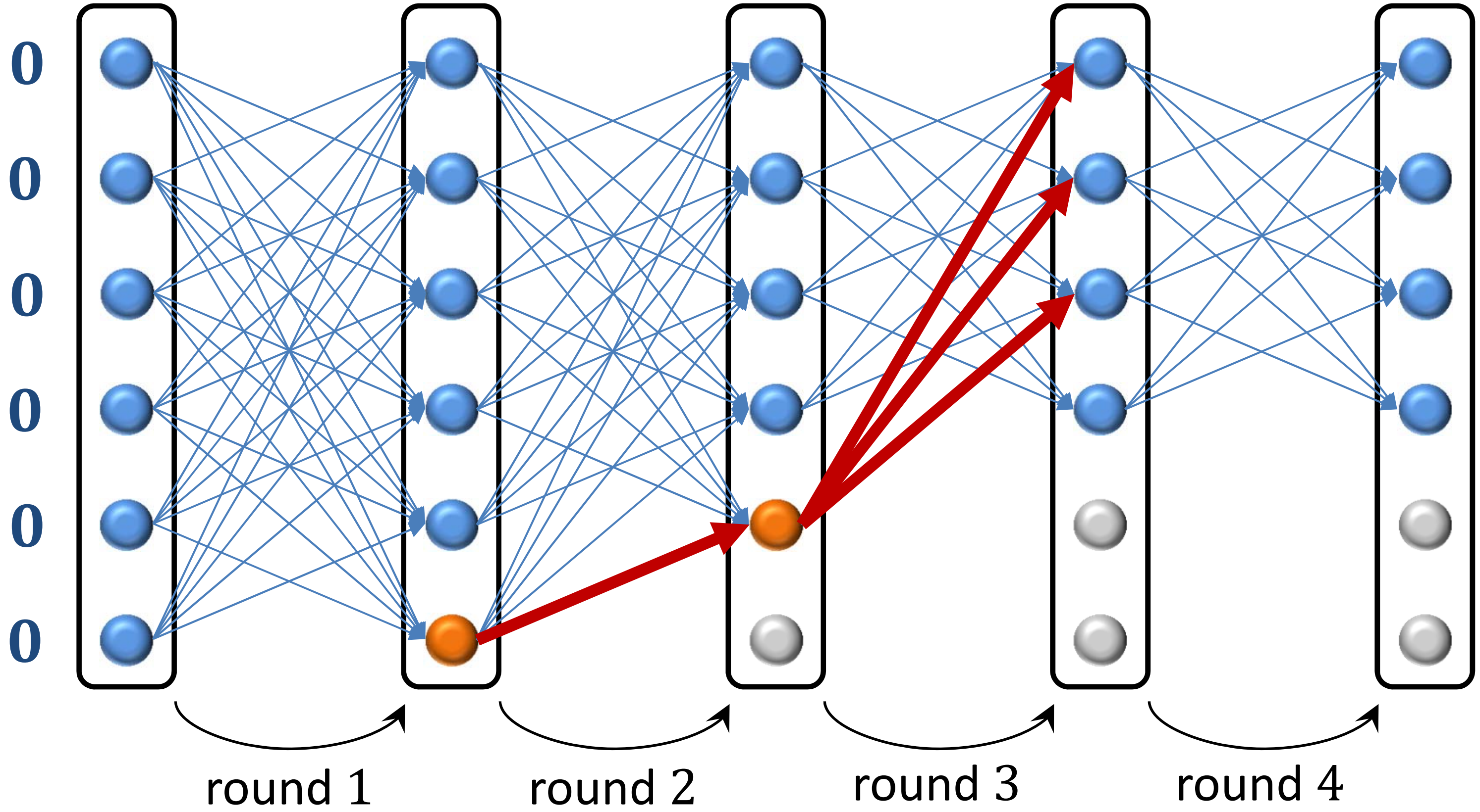

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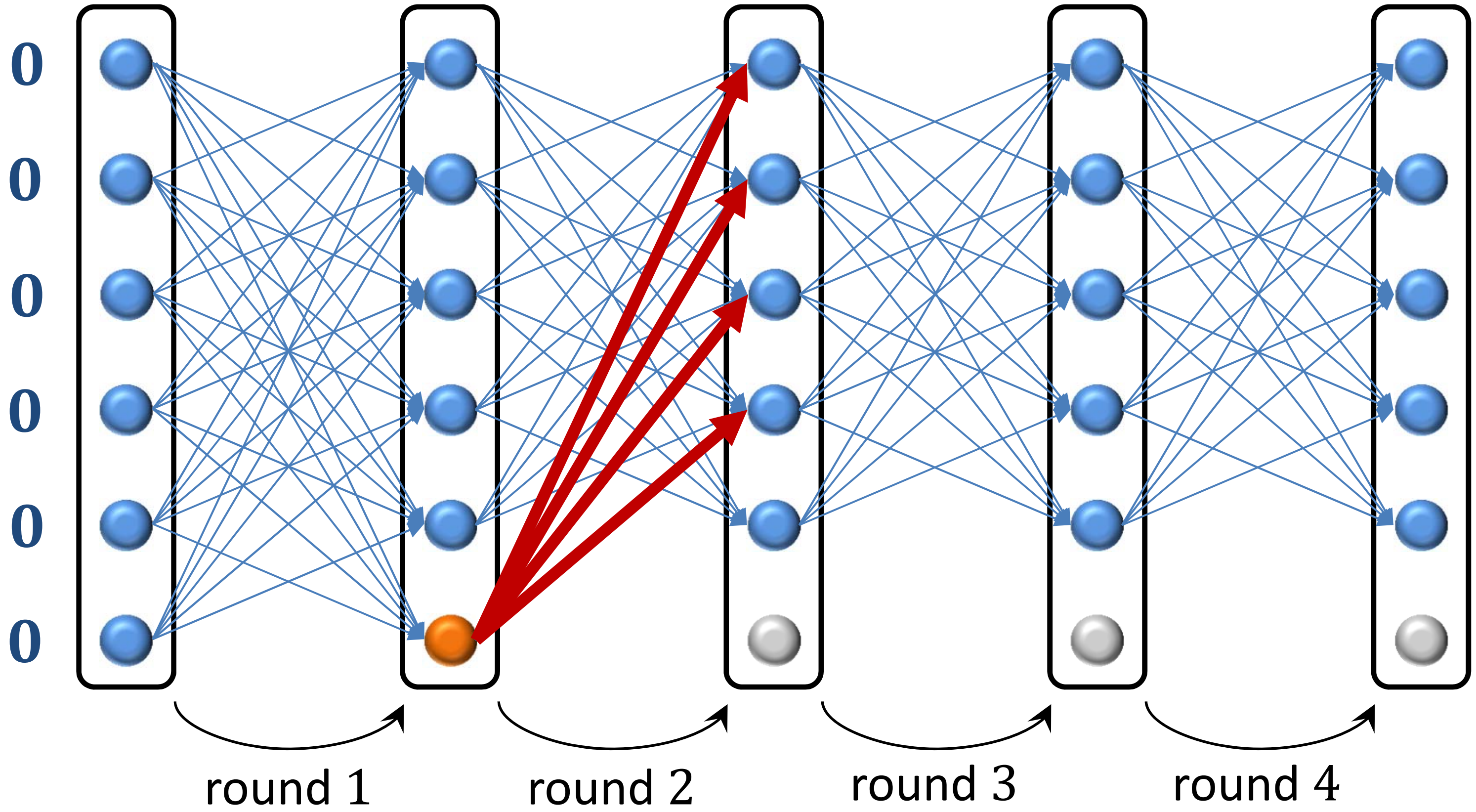

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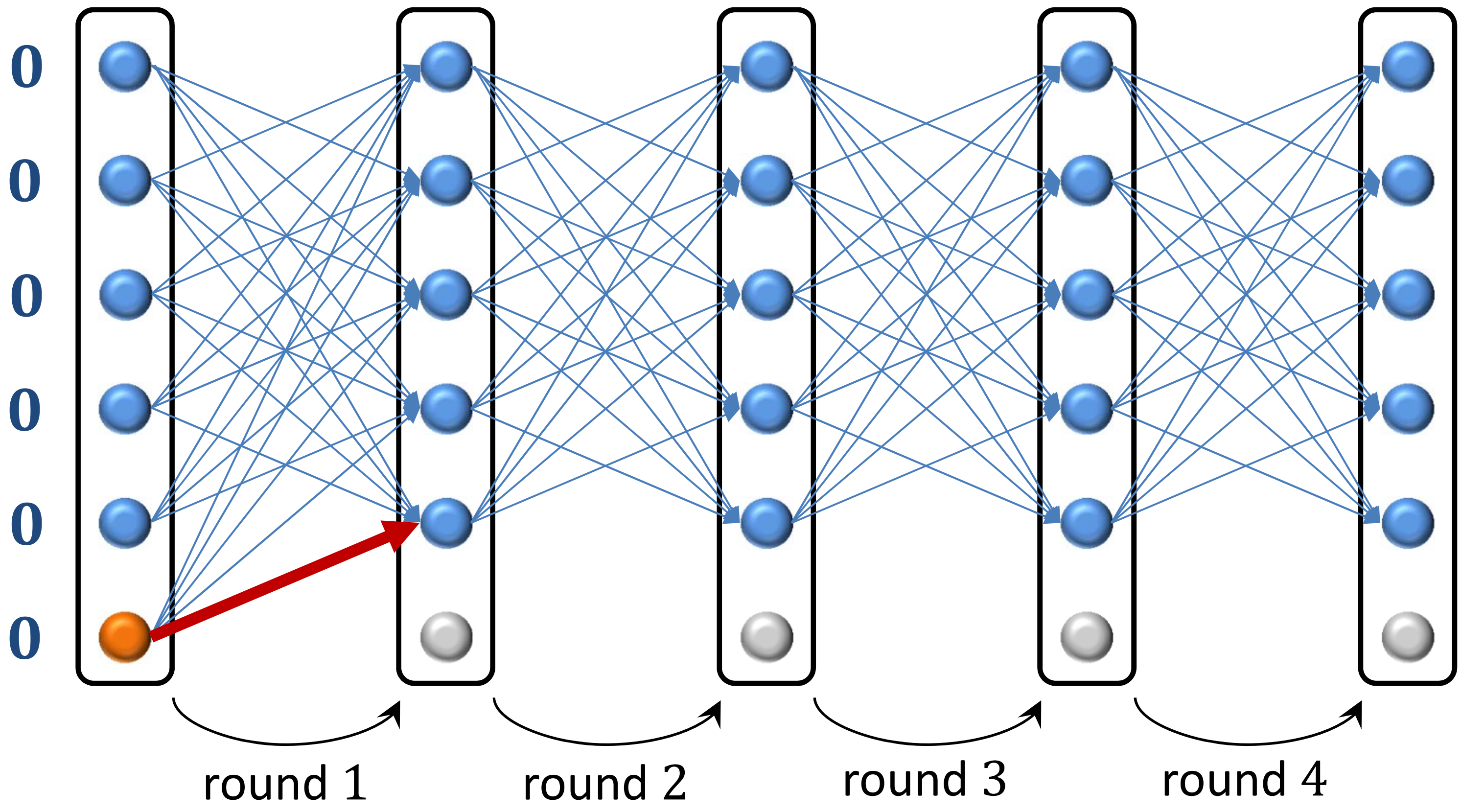

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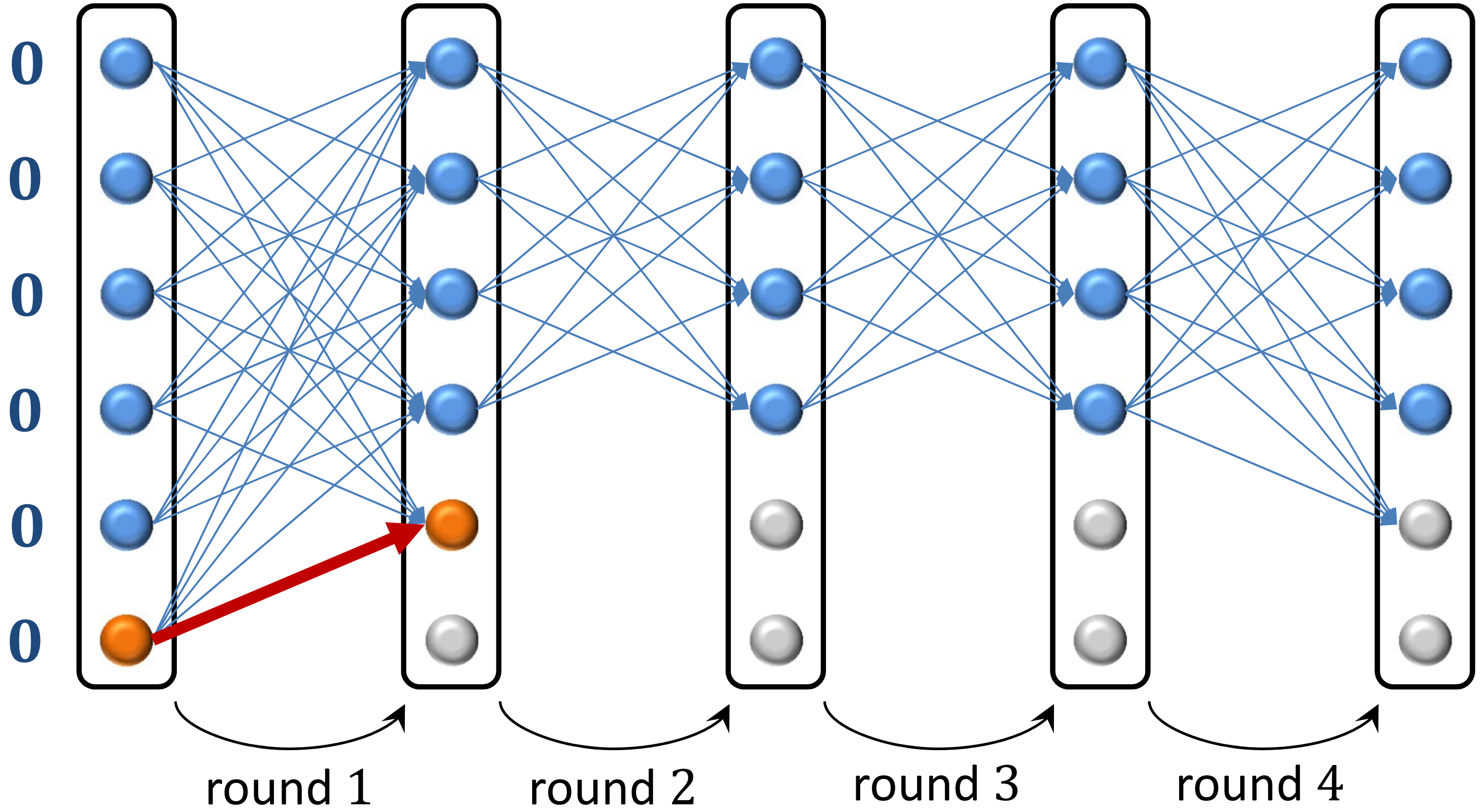

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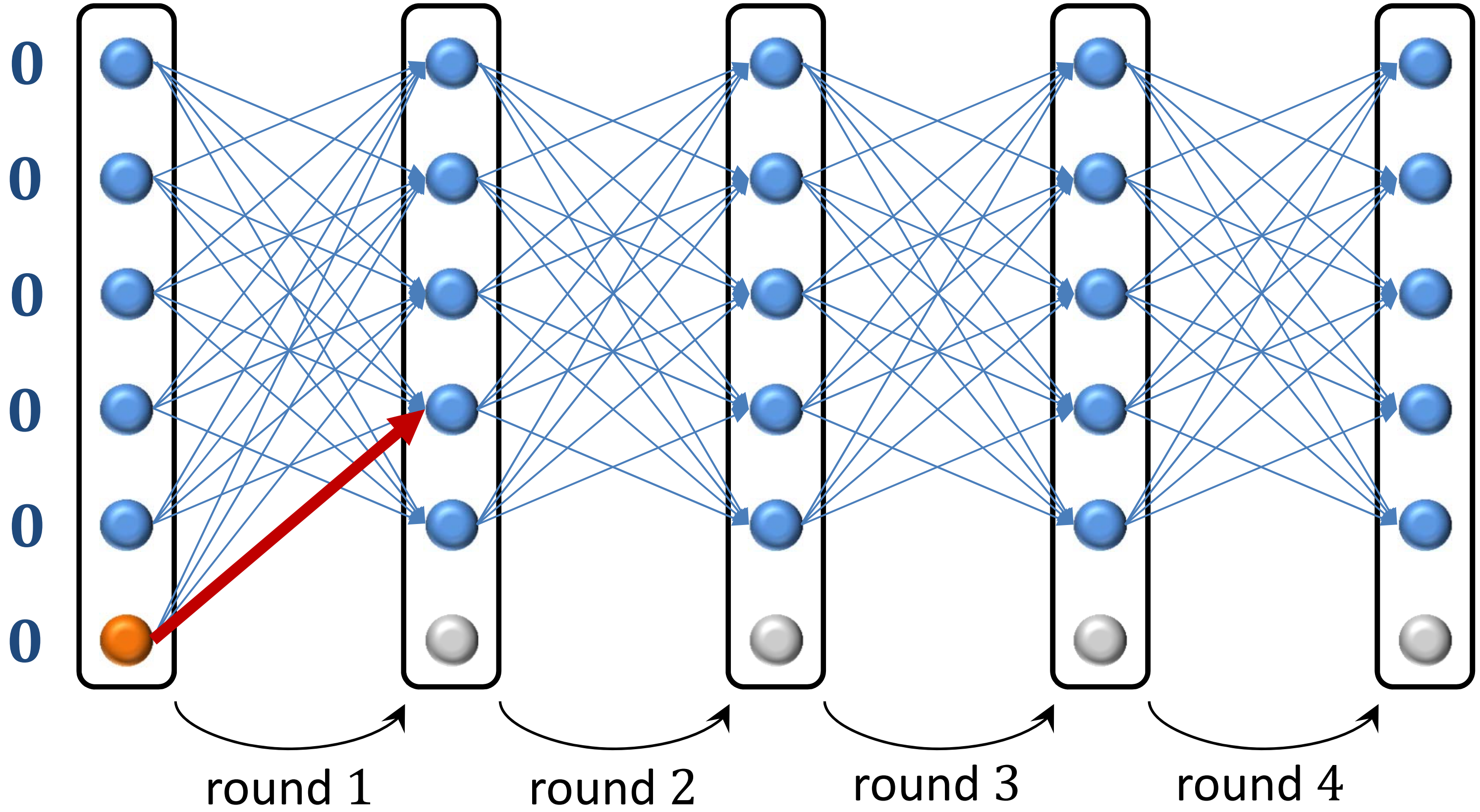

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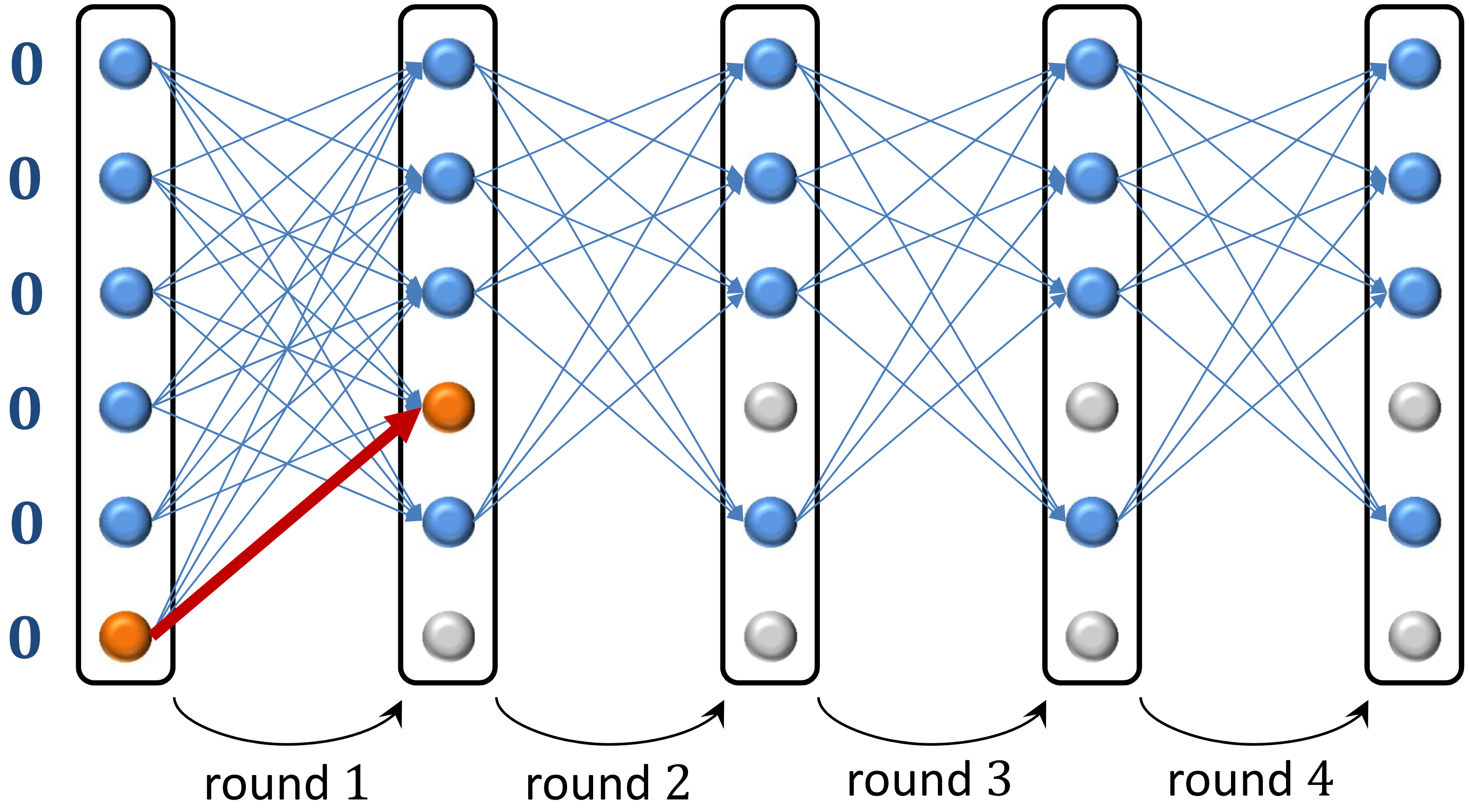

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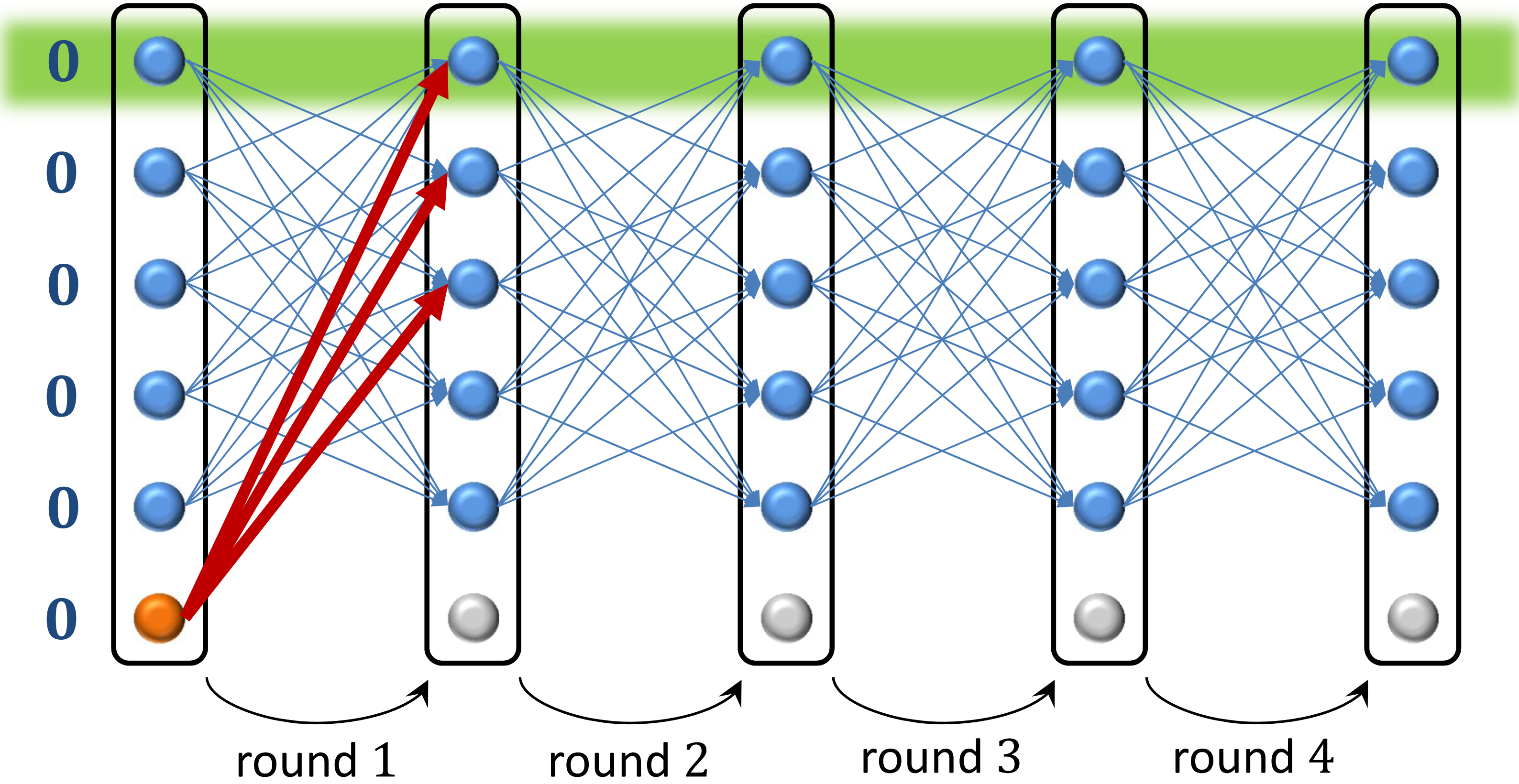

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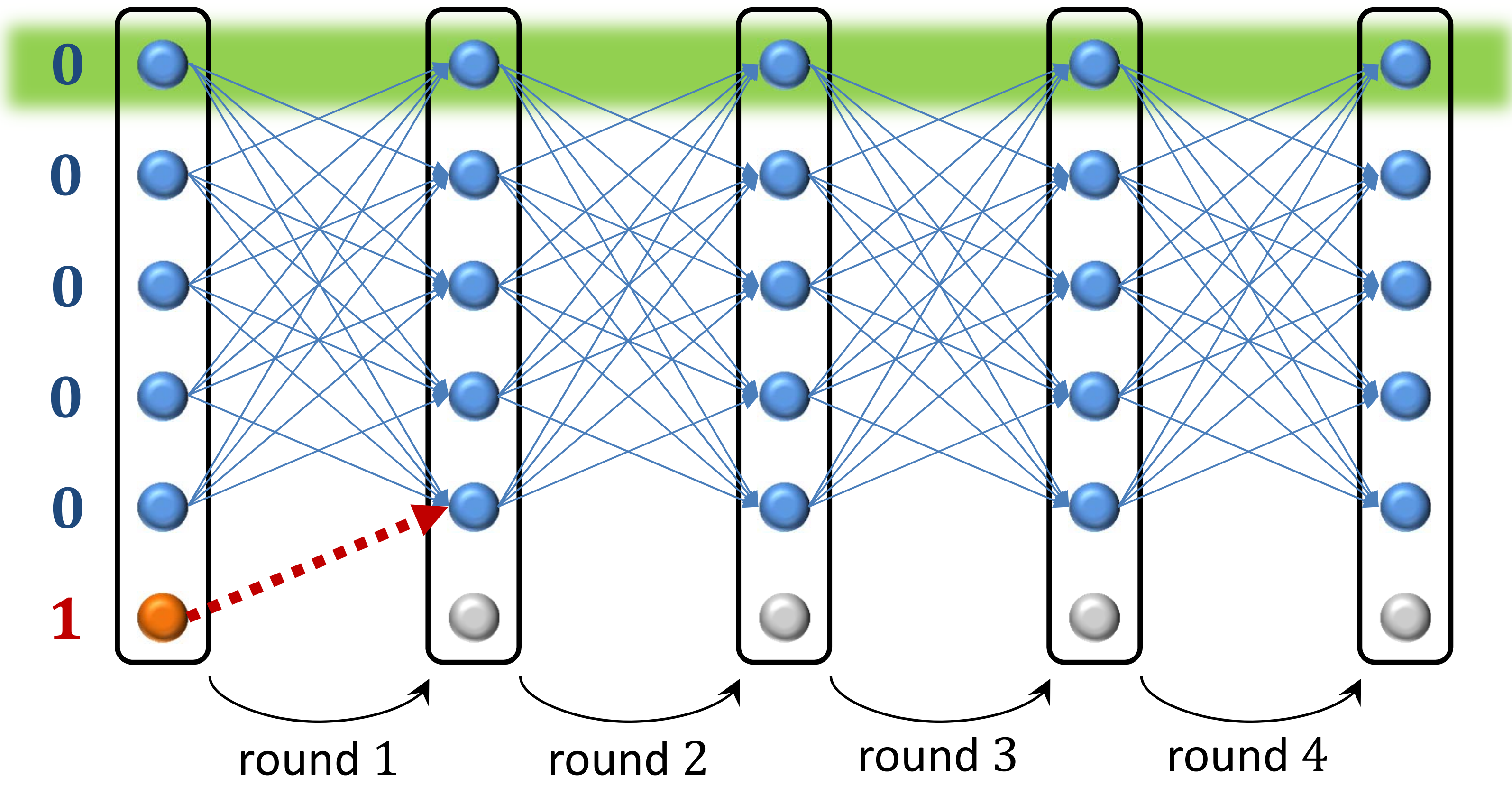

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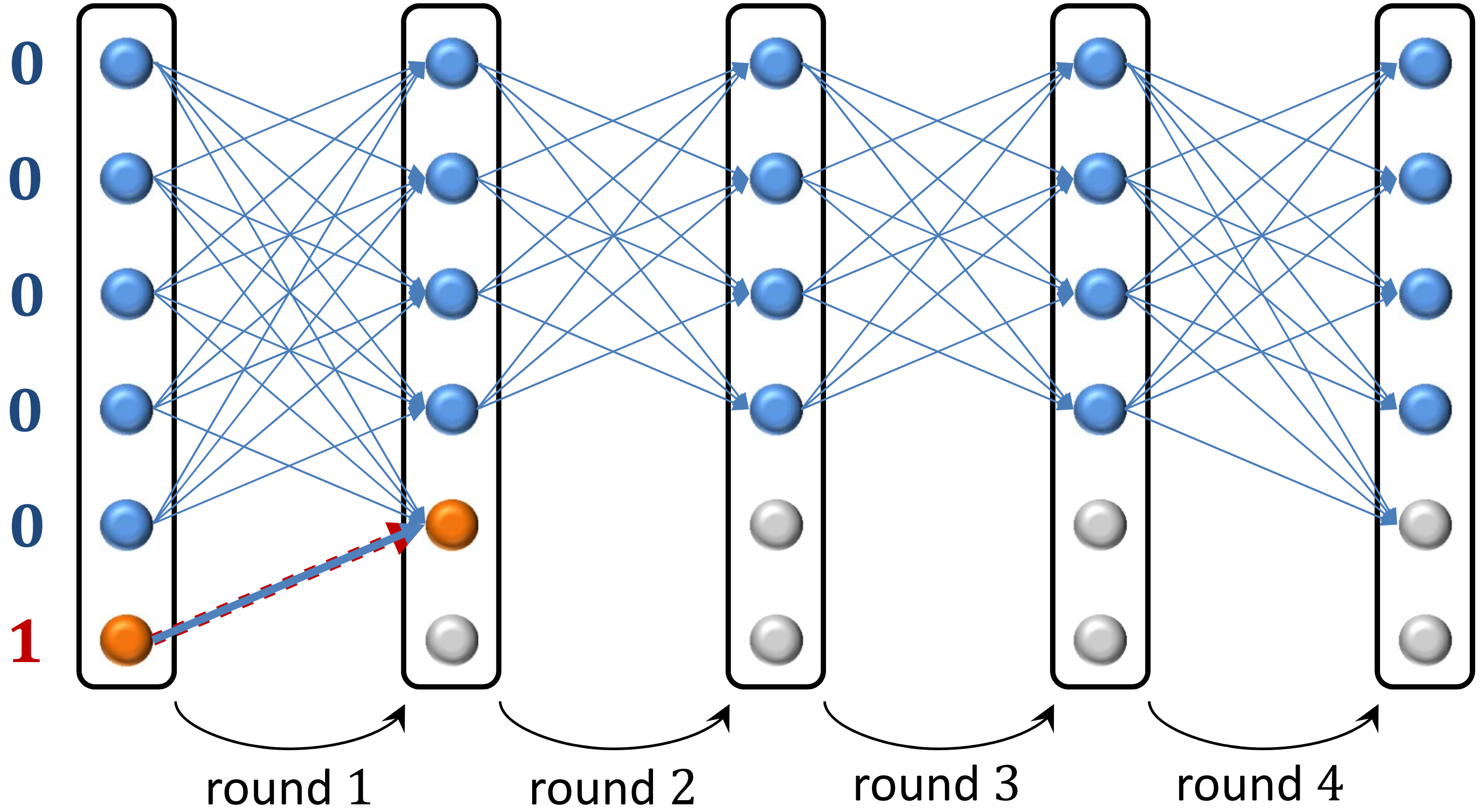

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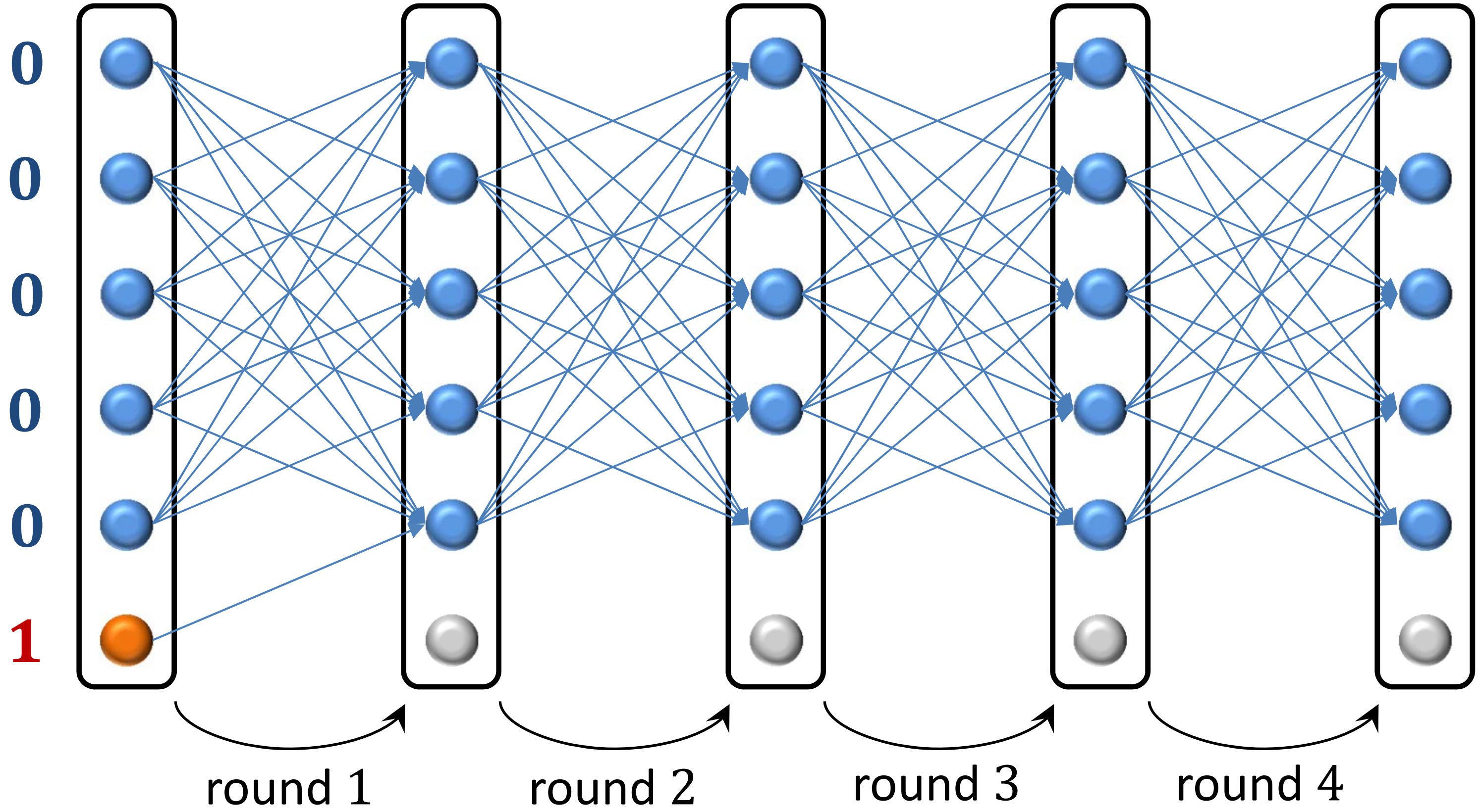

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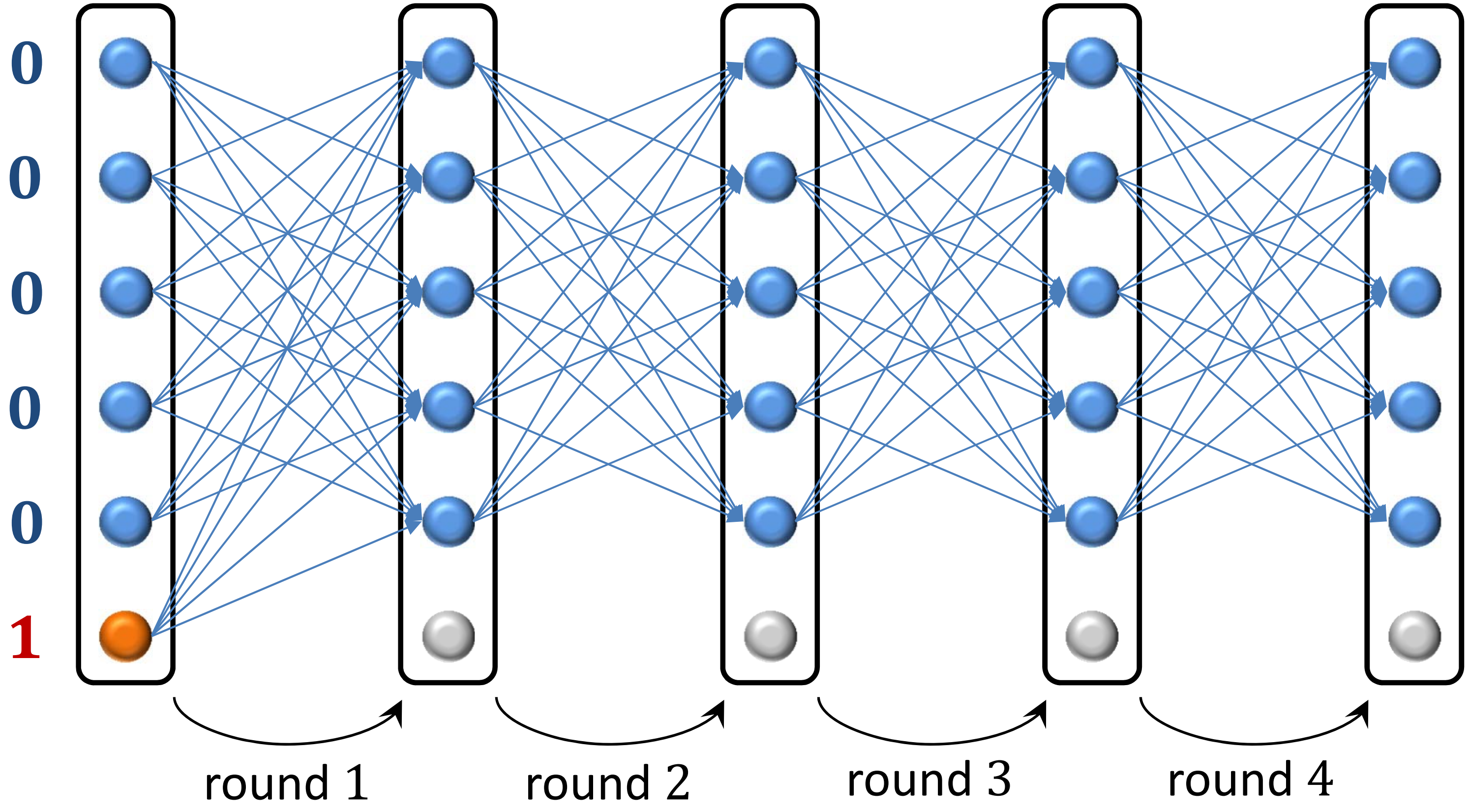

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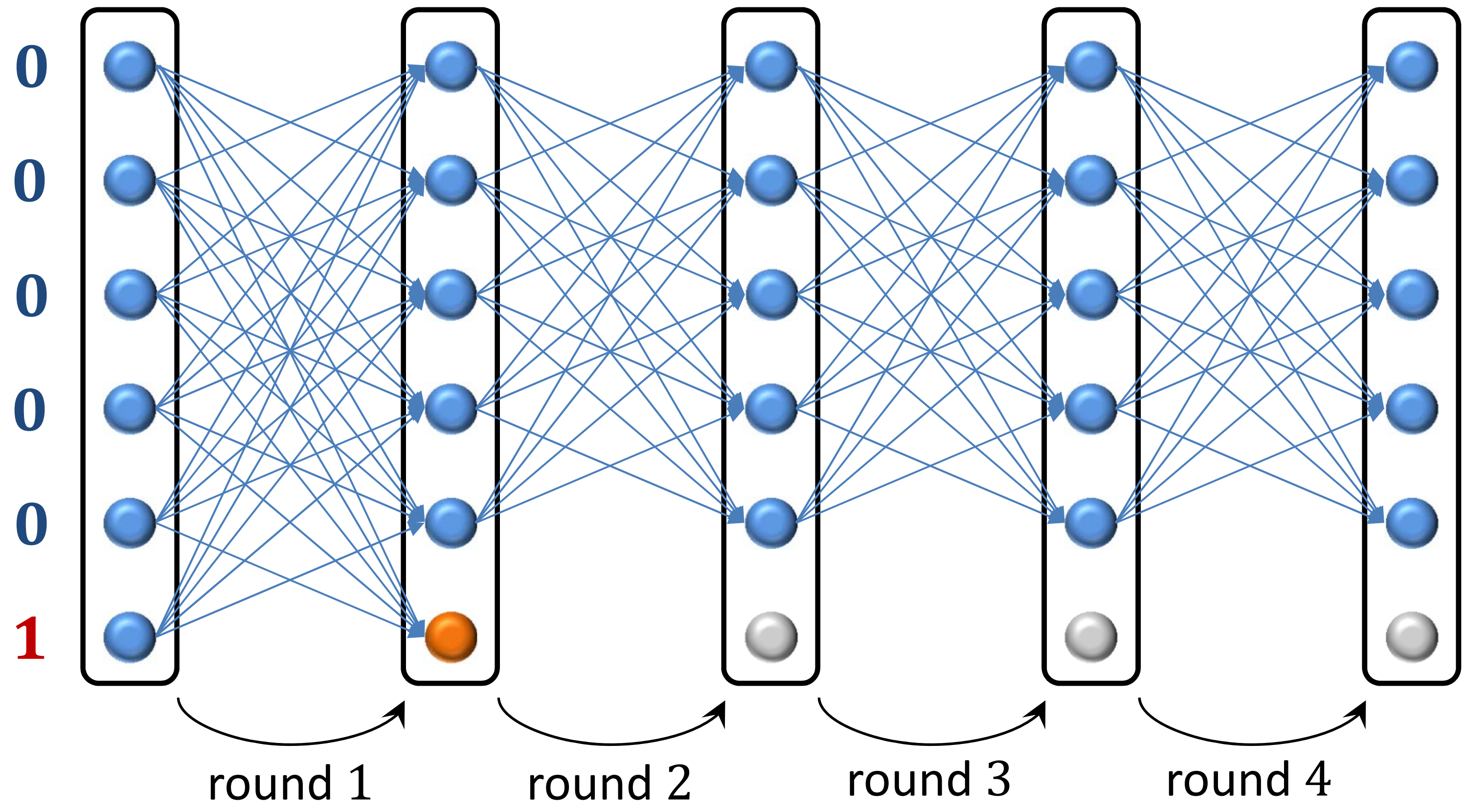

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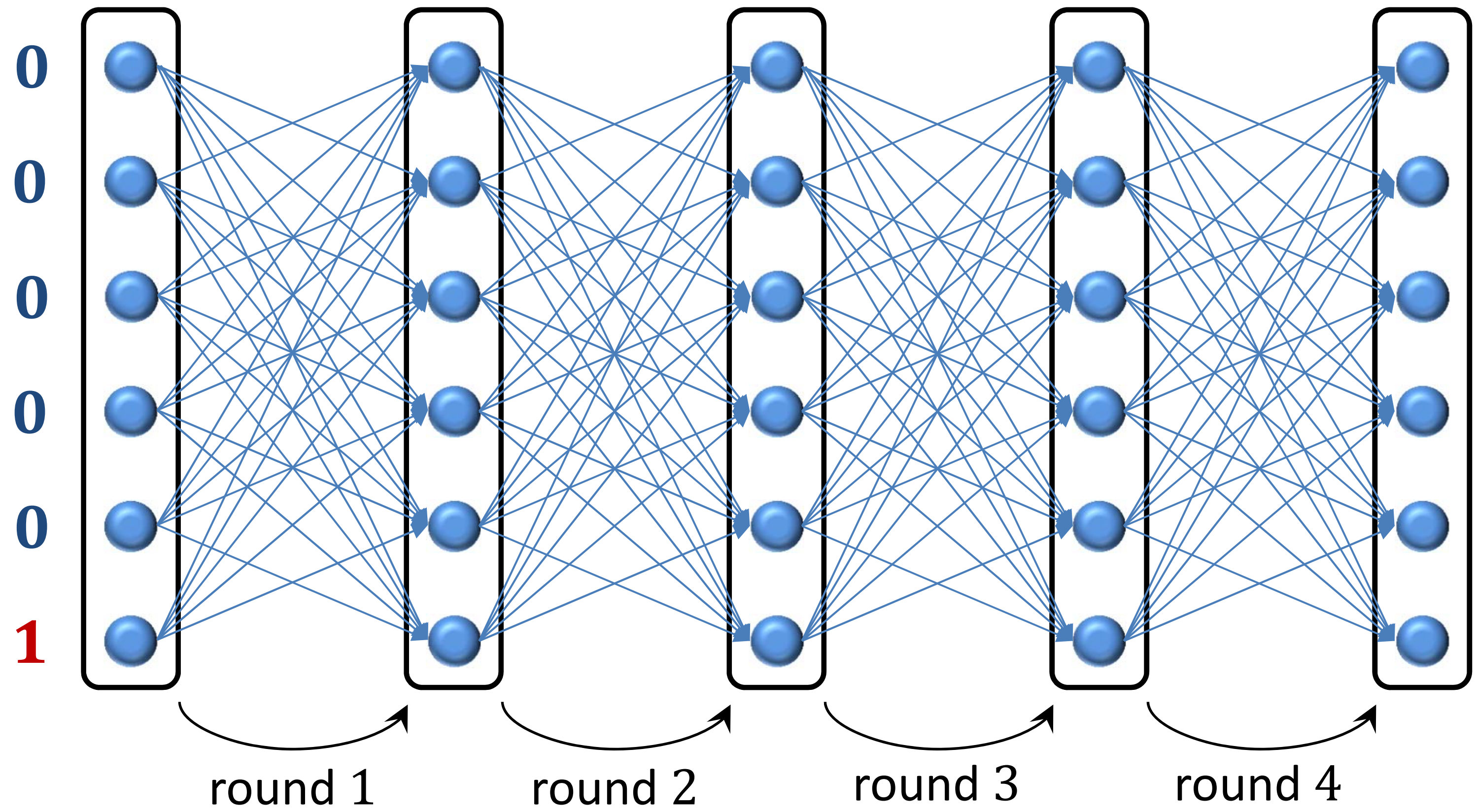

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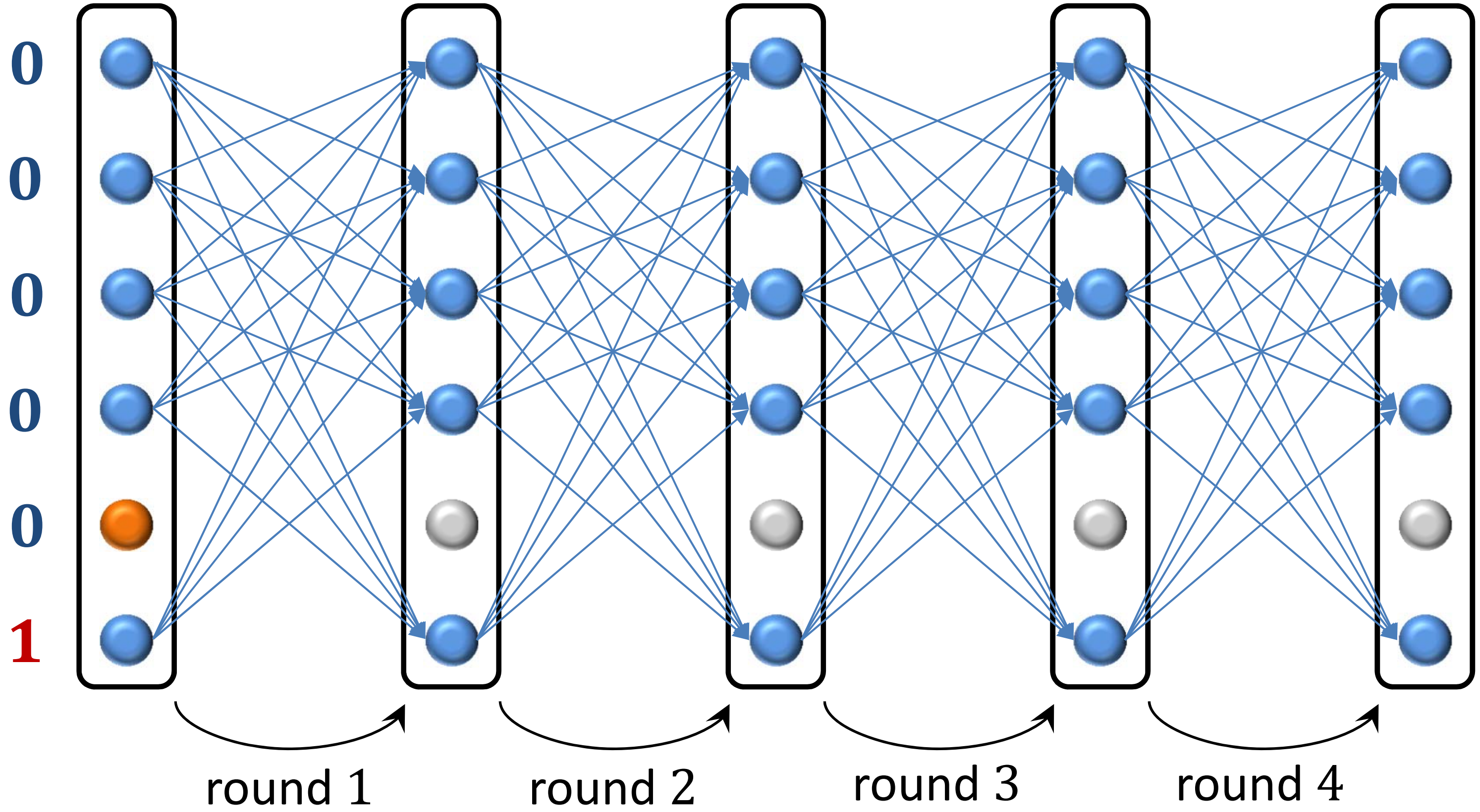

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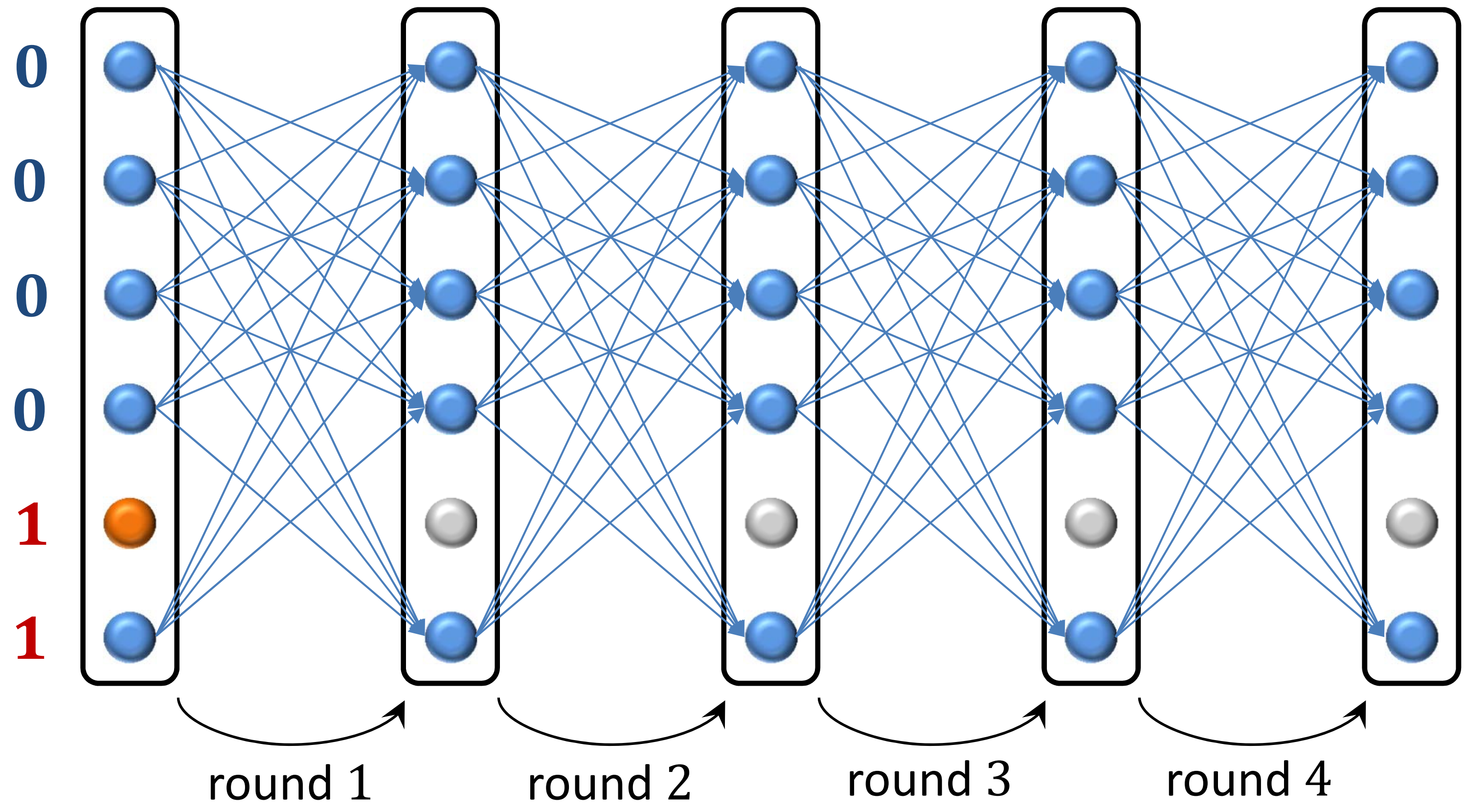

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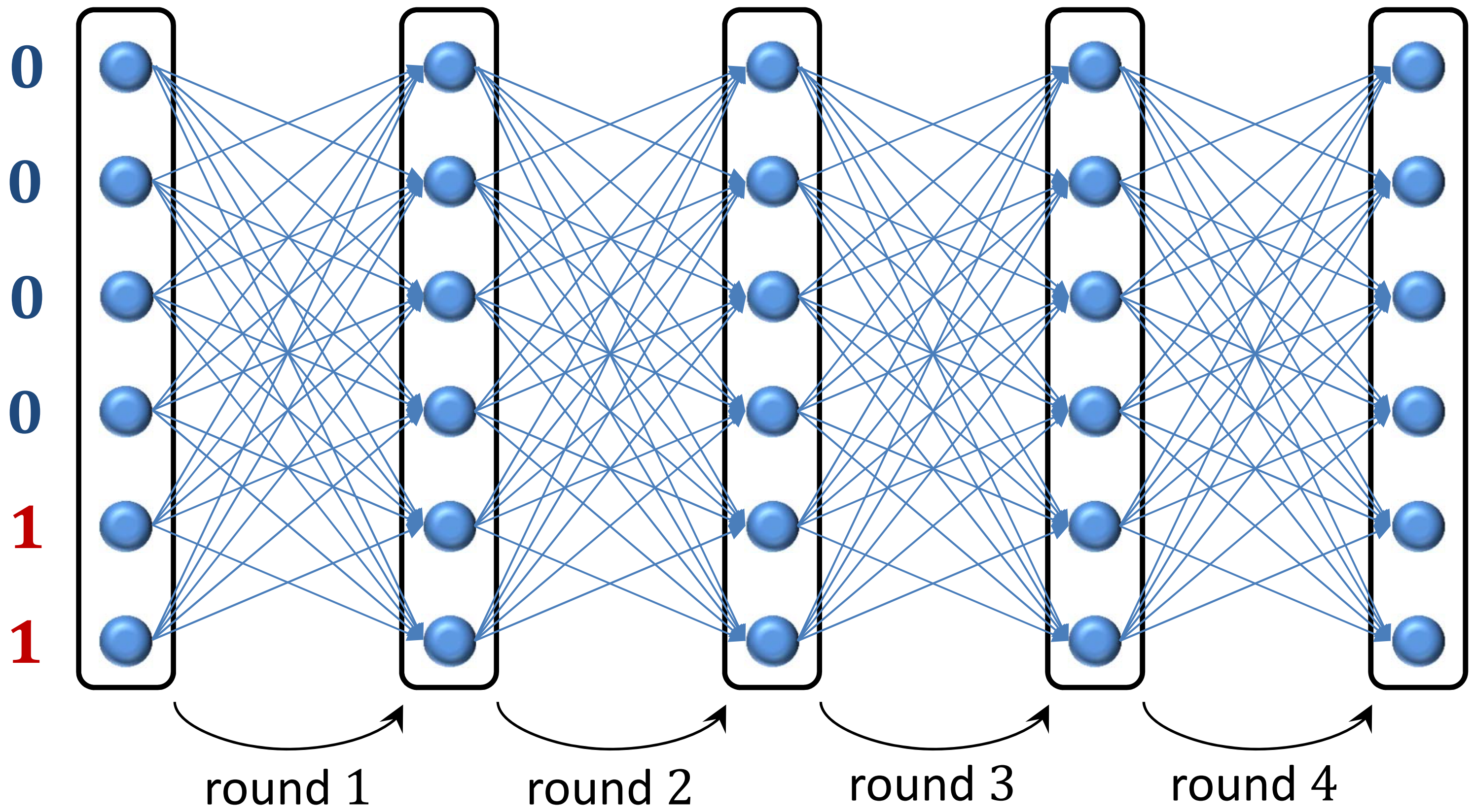

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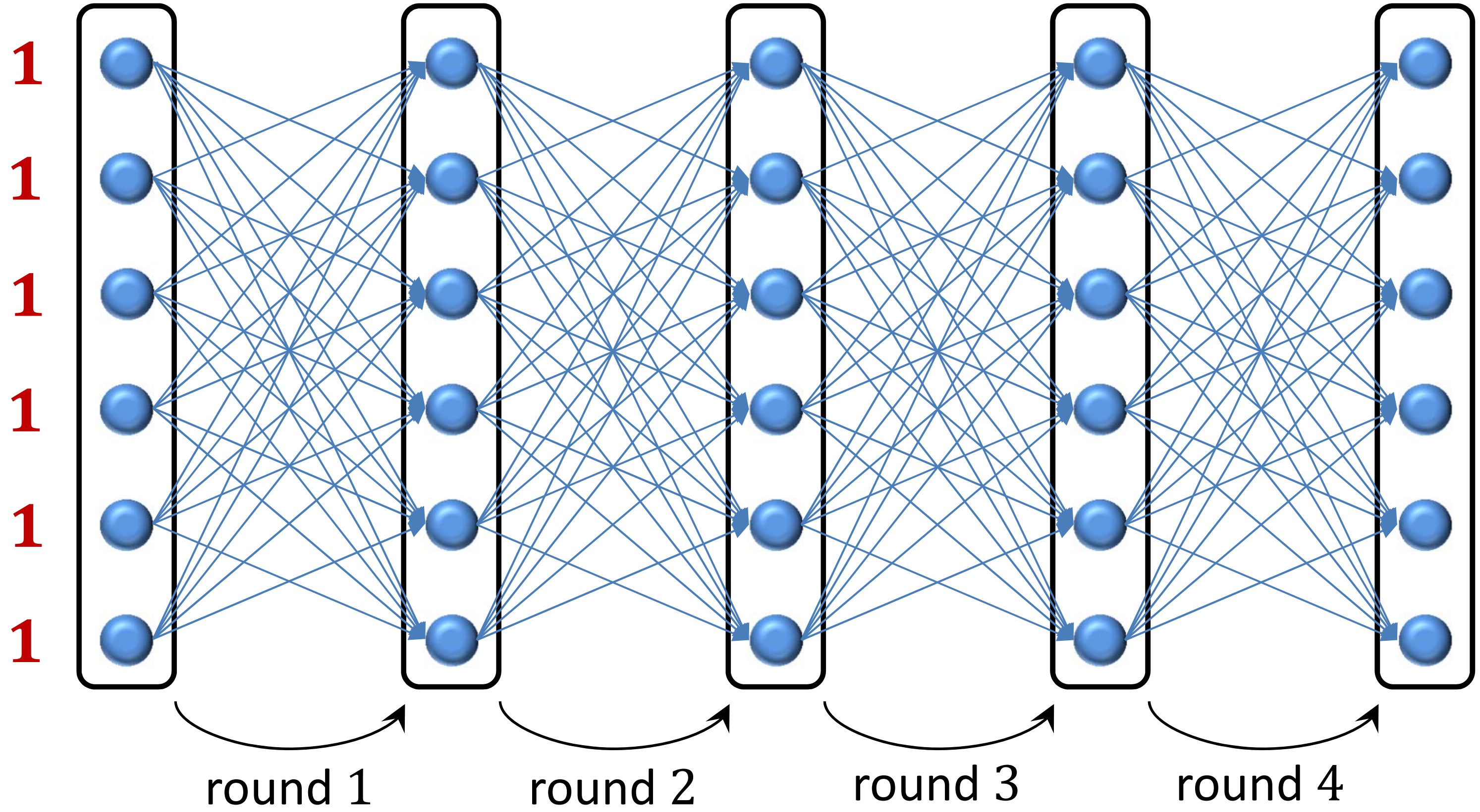


Lower Bound on Rounds

Theorem

If at most $f \leq n-2$ of n nodes of a synchronous message passing system can crash, at least $f+1$ rounds are needed to solve consensus.

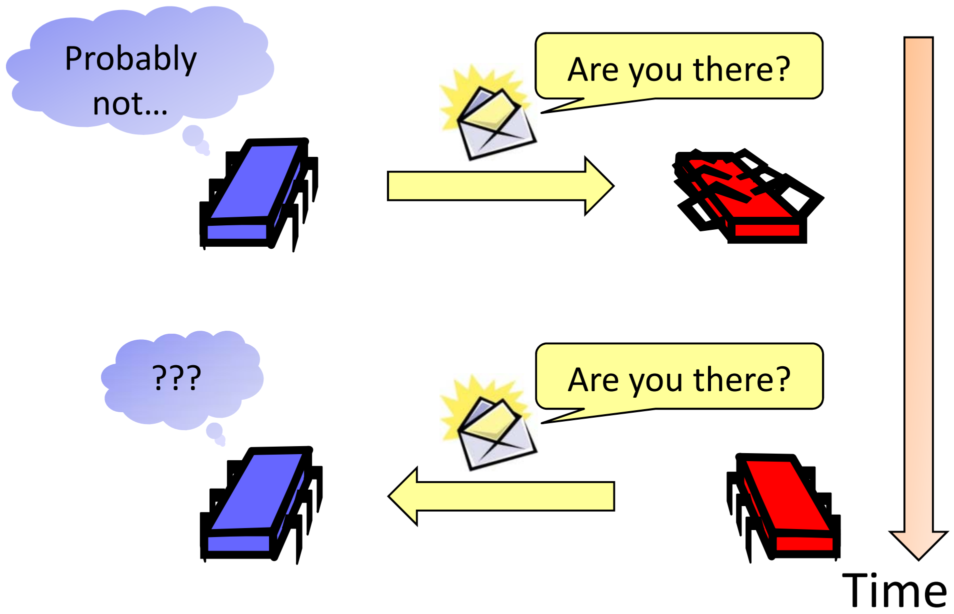
Proof:

- Similarity chain starting with fault-free all-zeroes execution and ending with fault-free all-ones execution
- In all executions, at most one crash per round
- Construction works as long as there are at least 2 non-faulty nodes in each execution ($n \geq f+2$)
- **Validity:** all-zeroes $\Longrightarrow$ decision 0; all-ones $\Longrightarrow$ decision 1
  Similarity Chain: same decision in all executions

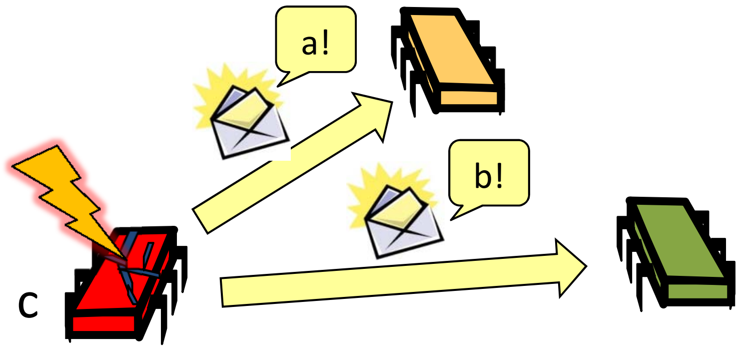
Arbitrary Behavior

- The assumption that processes crash and stop forever is sometimes too optimistic

- Maybe the processes fail and recover:

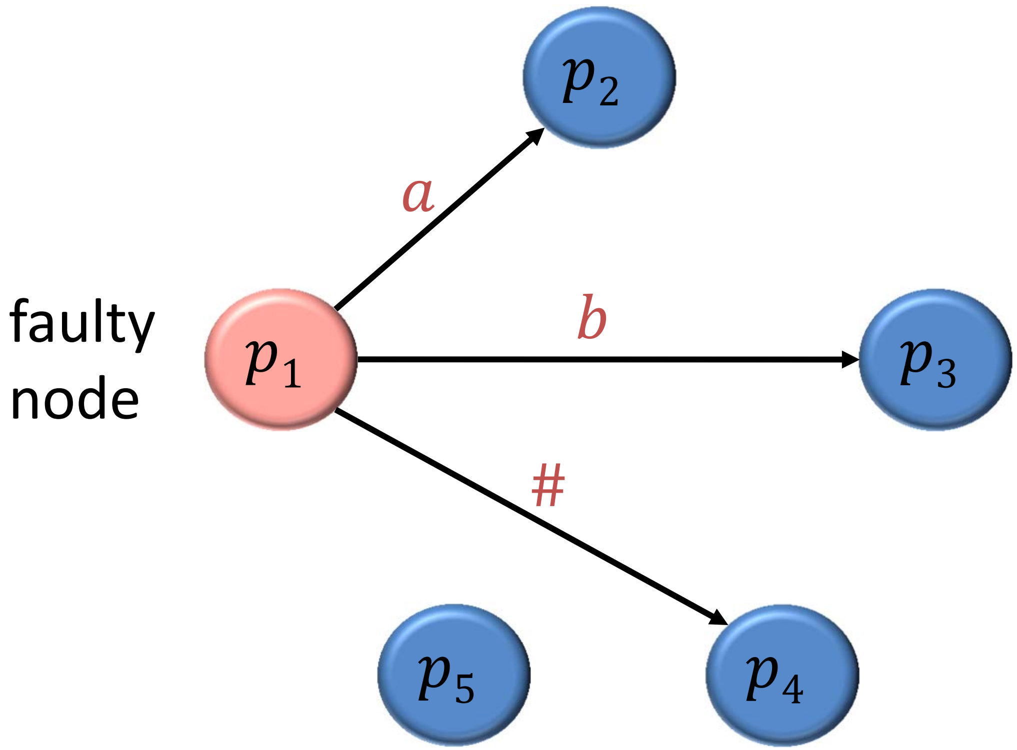


- Maybe the processes are damaged:

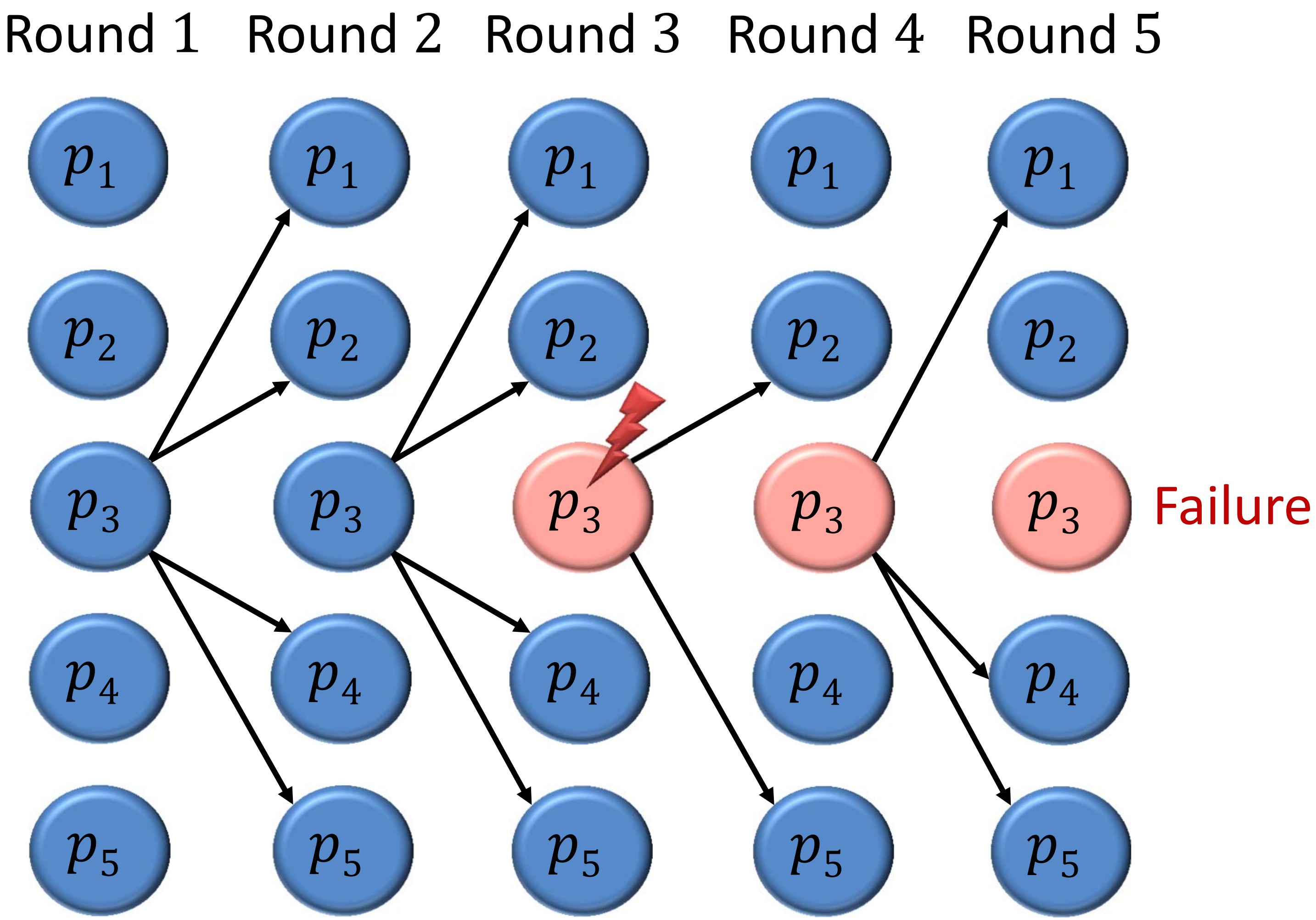


Consensus #5: Byzantine Failures

- Different processes may receive different values
- A Byzantine process can behave like a crash-failed process



After Failure, Node Remains in Network



Consensus with Byzantine Failures

- Again: If an algorithm solves consensus for f failed processes, we say it is an f-resilient consensus algorithm

- **Validity:** If all non-faulty processes start with the same value, then all non-faulty processes decide on that value
  - Note that in general this validity condition does not guarantee that the final value is an input value of a non-Byzantine process
  - However, if the input is binary, then the validity condition ensures that processes decide on a value that at least one non-Byzantine process had initially

- Obviously, any f-resilient consensus algorithm requires at least $f + 1$ rounds (follows from the crash failure lower bound)
- How large can f be…? Can we reach consensus as long as the majority of processes is correct (non-Byzantine)?

Impossibility

Theorem

There is no f-resilient Byzantine consensus algorithm for n nodes for $f \geq n/3$

Proof outline

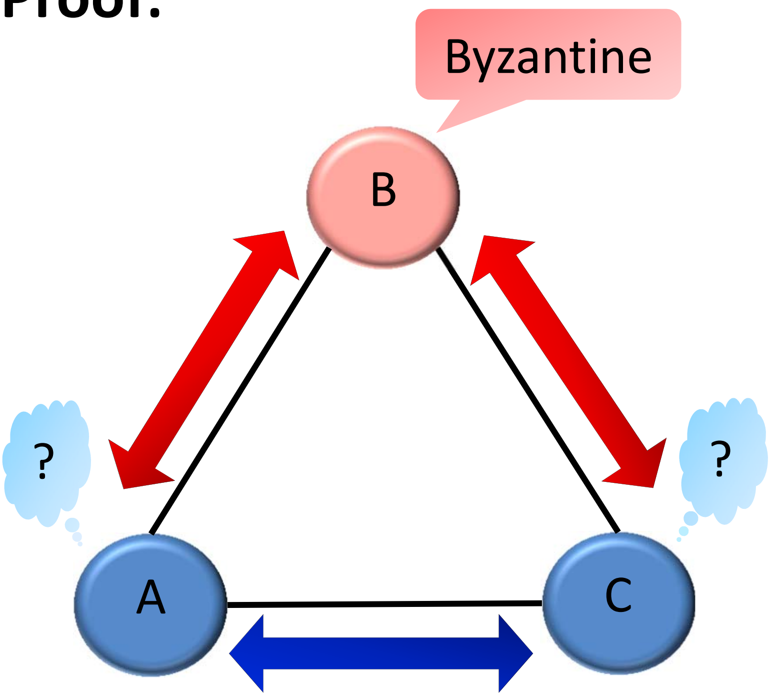
- First, we prove the 3 node case
  - not possible for $f = 1$
- The general case can then be proved by reduction from the 3 node case
  - Given an algorithm for n node and f faults for $f \geq n/3$, we can construct a 1-resilient 3-node algorithm

The 3 Node Case

Lemma

There is no 1-resilient algorithm for 3 nodes

Proof:

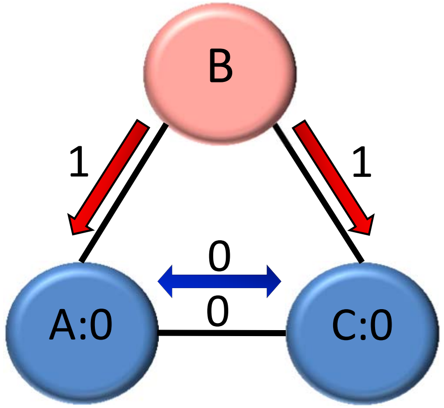


Intuition:

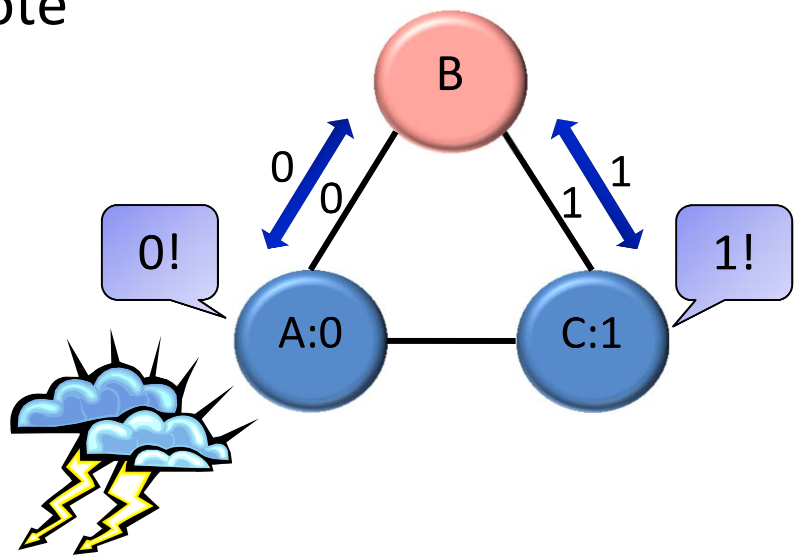
- Node A may also receive information from C about B's messages to C
- Node A may receive conflicting information about B from C and about C from B (the same for C!)
- It is impossible for A and C to decide which information to base their decision on!

Proof Sketch

- Assume that both A and C have input 0. If they decided 1, they could violate the validity condition → A and C must decide 0 independent of what B says
- Similary, A and C must decide 1 if their inputs are 1
- We see that the processes must base their decision on the majority vote

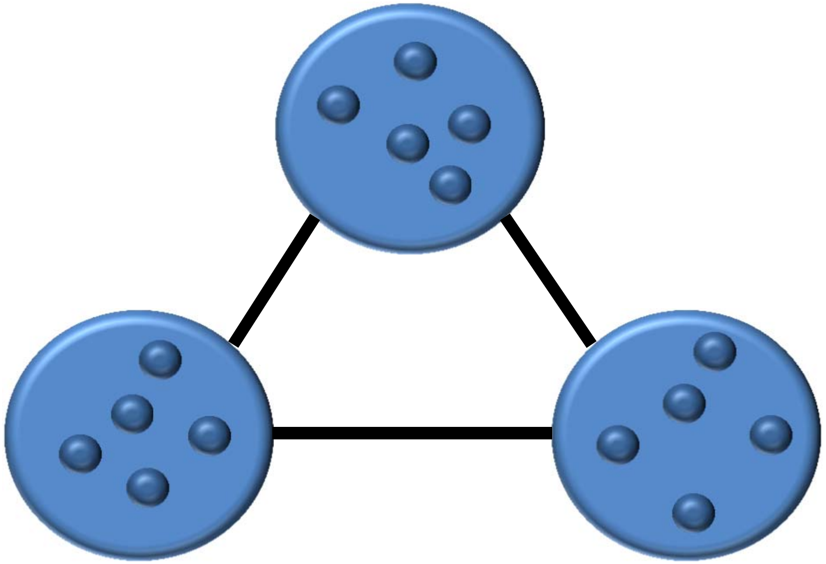


- If A's input is 0 and B tells A that its input is 0 → A decides 0
- If C's input is 1 and B tells C that its input is 1 → C decides 1



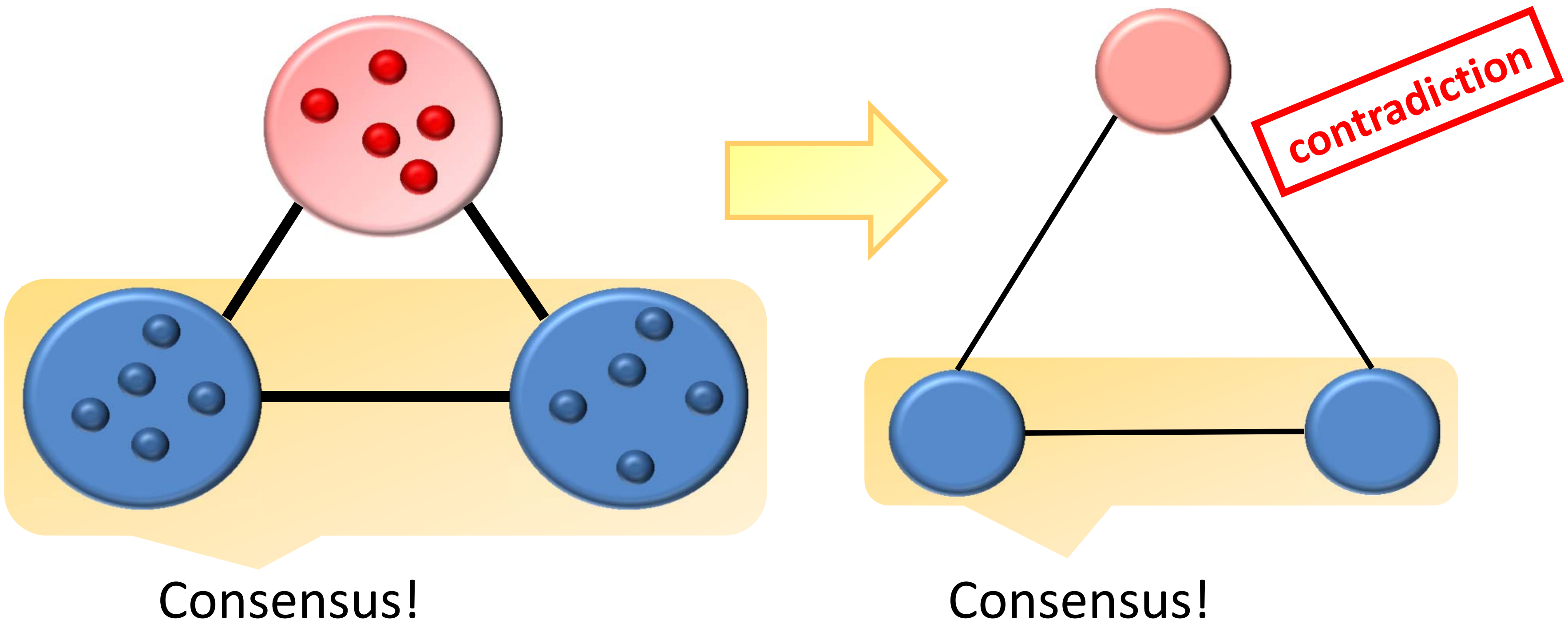
The General Case

- Assume for contradiction that there is an f-resilient algorithm A for n nodes, where $f \geq n/3$
- We use this algorithm to solve consensus for 3 nodes where one node is Byzantine!
- For simplicity assume that n is divisible by 3
- We let each of the three processes simulate $n/3$ processes



The General Case

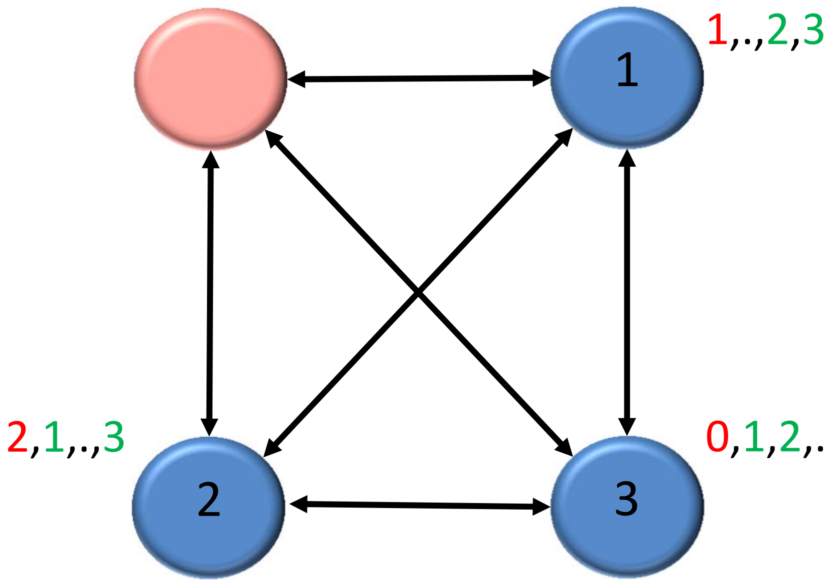
- One of the 3 nodes is Byzantine $\Longrightarrow$ its $n/3$ simulated nodes may all behave like Byzantine nodes
- Since algorithm A tolerates $n/3$ Byzantine failures, it can still reach consensus
  $\Longrightarrow$ We solved the consensus problem for three processes!



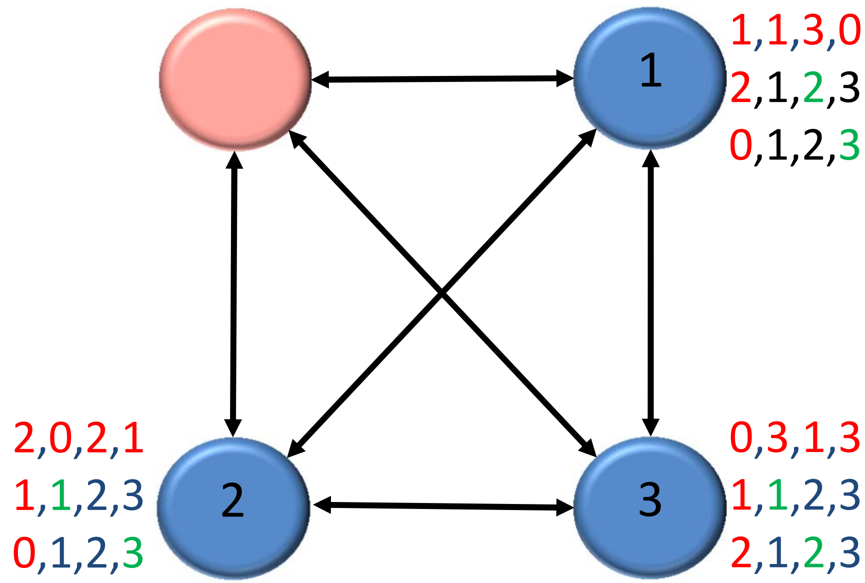
Cons. #6: Simple Byzantine Agreement Alg.

- Can the nodes reach consensus if $n > 3f$?
- A simpler question: What if $n = 4$ and $f = 1$?
- The answer is yes. It takes two rounds:

Round 1: Exchange all values



Round 2: Exchange received info



[matrix: one column for each original value, one row for each neighbor]

Simple Byzantine Agreement Algorithm

- After round 2, each node has received 12 values, 3 for each of the 4 input values (columns). If at least 2 of the 3 values of a column are equal, this value is accepted, otherwise it is discarded.
  - Values of honest nodes are accepted

Simple Byzantine Agreement Algorithm

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  - The value of the Byzantine node is accepted iff it sends the same value to at least two nodes in the first round.

Simple Byzantine Agreement Algorithm

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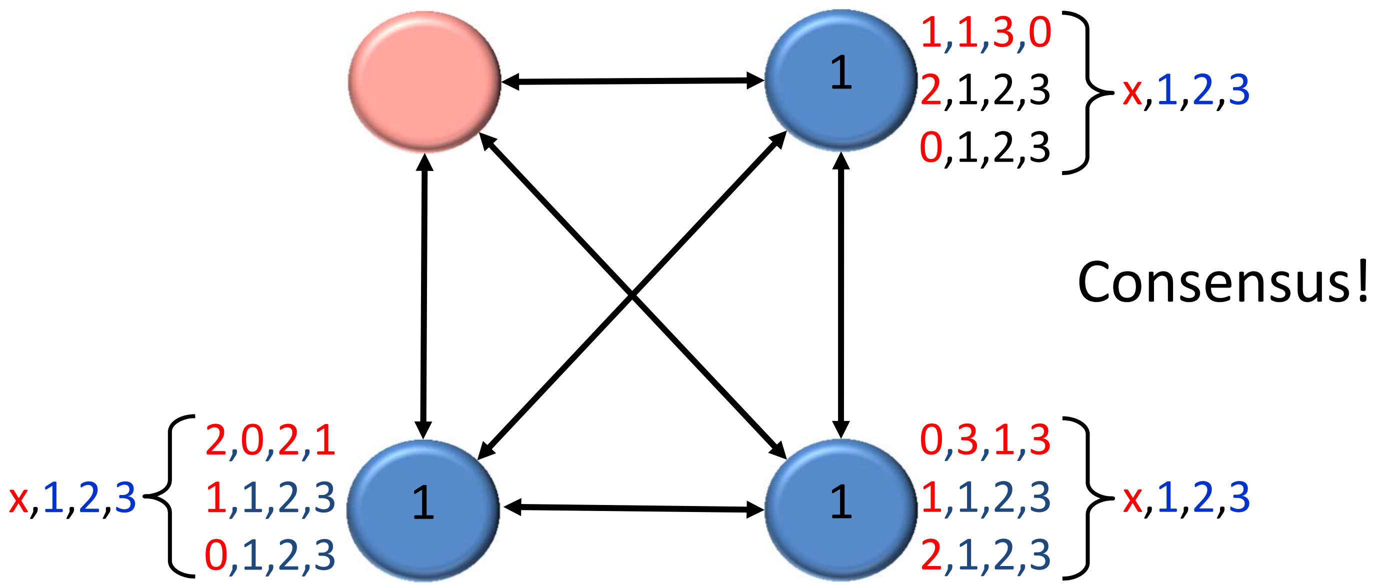
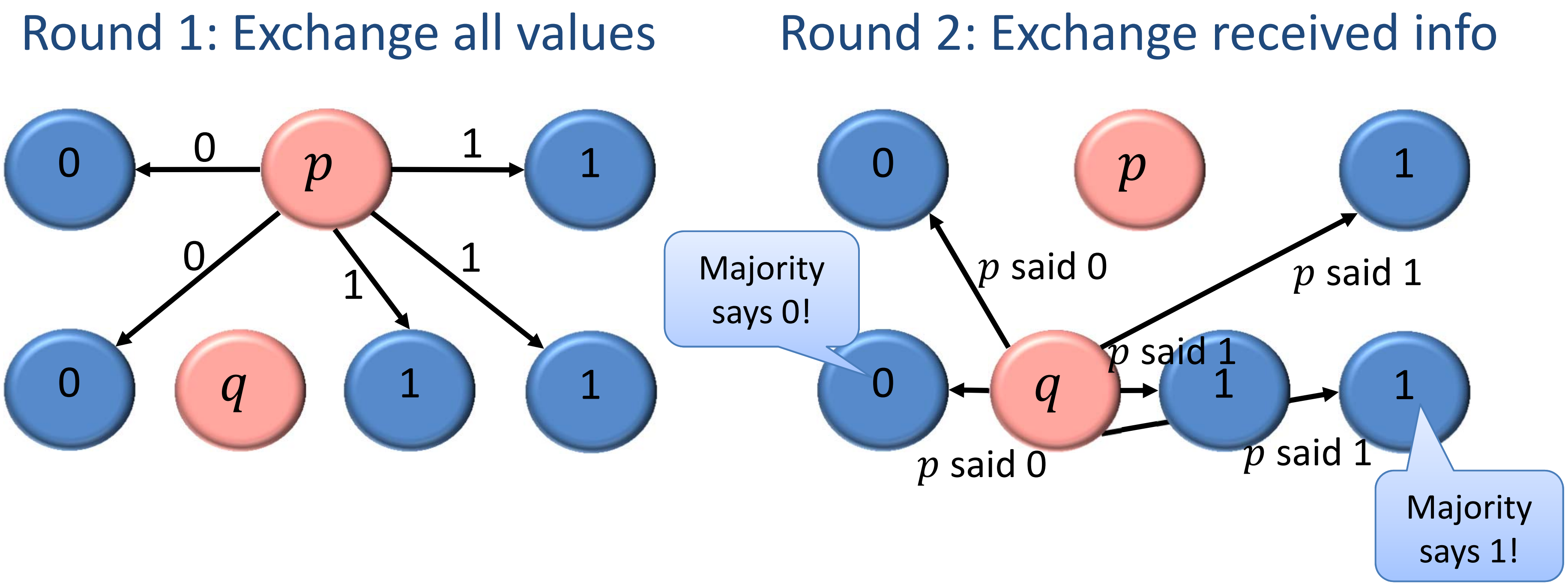
  - Values of honest nodes are accepted
  - The value of the Byzantine node is accepted iff it sends the same value to at least two nodes in the first round.
- Decide on minimum accepted value!



Simple Byzantine Agreement Algorithm

- Does the algorithm still work in general for any f and $n > 3f$?
- The answer is no. Try $f = 2$ and $n = 7$:



- The problem is that q can say different things about what p sent to q
  - What is the solution to this problem?

Simple Byzantine Agreement Algorithm

- The solution is simple: Again exchange all information!
- This way, the processes can learn that q gave inconsistent information about p
- Hence, q can be excluded, and also p if it also gave inconsistent information (about q).
- If $f = 2$ and $n > 6$, consensus can be reached in 3 rounds!
- In fact, the following algorithm solves the problem for any f and any $n > 3f$:

> Exchange all information for $f + 1$ rounds
> Ignore all processes that provided inconsistent information
> Let all processes decide based on the same input

Simple Byzantine Agreement Algorithm

The proposed algorithm has several advantages:

- \+ It works for any f and $n > 3f$, which is optimal
- \+ It only takes $f + 1$ rounds. This is even optimal for crash failures!
- \+ It works for any input and not just binary input

However, it has some considerable disadvantages:

- – "Ignoring all processes that provided inconsistent information" is not easy to formalize
- – The **size of the messages increases exponentially**!
  This is a severe problem. It is therefore worth studying whether it is possible to solve the problem with small(er) messages