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Network Algorithms, Summer Term 2018
Problem Set 9

hand in by Sunday, July 8, 2018

Exercise 1: Coloring Rings

In Chapter 1, we proved that a ring can be colored with 3 colors in $\log^* n + O(1)$ rounds. Clearly, a ring can only be (legally) colored with 2 colors if the number of nodes is even.

1. Prove that, even if the nodes in a directed ring know that the number of nodes is even, coloring the ring with 2 colors requires $\Omega(n)$ rounds![1]
2. Since coloring a ring with 2 colors apparently takes a long time, we again resort to the problem of coloring rings using 3 colors. Assume that a *maximal independent set* (MIS) has already been constructed on the ring, i.e., each node knows whether it is in the independent set or not. Give an algorithm to color the ring with 3 colors in this scenario! What is the time complexity of your algorithm? Deduce from this a lower bound for computing a MIS!

Exercise 2: Lower bound for sinkless orientation

In this exercise we look at another lower bound, for a problem called *sinkless orientation.* Given an unoriented graph $G = (V, E)$, an orientation maps each edge $\{u, v\} \in E$ to a directed edge $s(\{u, v\}) \in \{(u, v), (v, u)\}$ (read (u, v) as oriented from u to v and *outgoing* for u). An orientation is *sinkless* if every node has an outgoing edge.

Consider the following model, based on the LOCAL model.

1. Nodes do not have unique names, but have a node 2-coloring.
2. Edges are colored with $\Delta + 1$ colors.
3. The input graph is an infinite 3-regular tree.

Let $N_t[v]$ denote the full t-hop neigbhorhood of node v (edges are included only if both endpoints are included). For an edge $\{u, v\}$, let $N_t[u, v] = N_t[u] \cup N_t[v]$.

We want to show that there is no T such that sinkless orientation can be computed in time T in this setting. The strategy is to show that a T-time algorithm implies a $(T - 1)$-time algorithm.

a) Consider a T-time algorithm A for sinkless orientation. Show that the output of edge $e = \{u, v\}$ only depends on $N_T[u] \cap N_T[v]$.

This implies that we can consider algorithms that look at the neighborhoods of edges. We say that an algorithm has running time T if it looks at the T-edge neighborhood $N_T[e]$ of each edge e.

[1] As in the lecture, the message size and local computations are unbounded and all nodes have unique identifiers from 1 to n.

b) Consider an edge $e = \{u, v\}$, and edges $\{u, w\}$ and $\{v, w'\}$. Show that given $N_{T-1}[e]$, the outputs of $\{u, w\}$ and $\{v, w'\}$ are independent from each other.
   Hint: show that the inputs that determine the output of $\{u, w\}$ are disjoint from the inputs that determine the output of $\{v, w'\}$.
c) Assume A_T is a T-time edge algorithm for sinkless orientation. Show that all edges of color $\Delta + 1$ can be decided in time $T - 1$.
   Hint: use the observation from b). When can u and v become sinks?
d) Show that an algorithm that runs in time T implies an algorithm that runs in time $T - 1$.
e) Show that sinkless orientation cannot be solved in time $T = 0$. Why does this, together with d), imply the claim that sinkless orientation cannot be solved in this setting?
f) There exist 3-regular graphs with girth (length of the shortest cycle) $\Theta(\log n)$. What does our argument imply in such graphs?
g) Why does our argument not work if the nodes have *unique* names?