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Theory of Distributed Systems
Exercise Sheet 7

Due: Wednesday, 21st of June 2023, 12:00 noon

Exercise 1: Maximal matching *(6 Points)*

In the following, we are given a graph $G = (V, E)$ of maximum degree Δ, where *nodes* are colored with c colors, and the goal is to produce a maximal matching. A maximal matching is a subset of edges $X \subseteq E$ satisfying the following:

- For all e_1, e_2 in X, it holds that e_1 and e_2 are not incident to the same node, that is, they do not share endpoints. Hence, for each node it holds that at most one incident edge is in the matching.
- Adding any additional edge of $E \setminus X$ to X would violate the above constraint.

Hence, we are interested in a subset of edges that are independent such that this subset cannot be extended.

1. Consider the case where $c = 2$, that is, the graph is bipartite and properly colored with two colors, black and white. Assume that nodes know the value of Δ and c. Show that maximal matching can be solved in $O(\Delta)$ rounds.
   Spoiler hint: see the footnote[1].
2. Assume that c and Δ are known to each node. Show that, for any value of c, this problem can be solved in $O(c\ \Delta)$.
3. Show that this problem can be solved in $O(c\ \Delta)$ even in the case where c and Δ are unknown to the nodes.

Exercise 2: Coloring planar graphs *(5 Points)*

Show how to color a planar graph with $O(1)$ colors in $O(\log n)$ time.
Hint: every planar graph satisfies that its average degree is less than 6, where the average degree of a graph G is defined to be the sum of all the degrees of the nodes in G divided by the total number of nodes in G. Use the same idea of the algorithm for unrooted trees presented in the lecture.

Exercise 3: Coloring unrooted trees *(5 Points)*

Show that it is possible to 3-color unrooted trees in $O(\log n)$ time.
Hint: modify the algorithm of 9-colors unrooted trees presented in the lecture.

[1] White nodes can try to „propose" to each black neighbor, by trying one neighbor at a time. Black nodes can accept the first proposal and reject all the others.

Exercise 4: Color Reduction *(4 Points)*

a) Given a graph which is colored with $m > \Delta + 1$ colors, describe a method to recolor the graph in one round using $m - \lfloor \frac{m}{\Delta+2} \rfloor$ colors. Assume Δ is known to the nodes.

   Hint: Partition the set of colors into sets of size $\Delta + 2$ (where only one of the sets might be of size less than $\Delta + 2$), and recall the color reduction method from the lecture.

b) Show that after $O(\Delta \log(m/\Delta))$ iterations of step a), one obtains a $O(\Delta)$ coloring.