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Theoretical Computer Science - Bridging Course
Exercise Sheet 1

Due: Tuesday, 23rd of April 2024, 12:00 pm

Exercise 1: Validness of Mathematical Induction *(Bonus Points)*

To prove that a statement $P(n)$ is true for all $n \in \mathbb{N}$, mathematical induction can be stated as

$$(P(1) \wedge \forall k(P(k) \Rightarrow P(k+1))) \Rightarrow \forall n P(n)$$

$P(1)$ stands for the *Base Case*, and $P(k) \Rightarrow P(k+1)$ for *Induction Hypothesis*. The statement above is valid. *i.e) if Antecedent is true, then the Consequent can't be false.* ,which justifies the use of Mathematical Induction in this case. Using Contradiction, prove the validity of mathematical induction. In other words, *Using contradiction, prove that if* $P(1) \wedge \forall k(P(k) \Rightarrow P(k+1))$ *is true, then* $\forall n P(n)$ *necessarily follows.*
Use the *Well-Ordering Property* of natural numbers to help finding a contradiction.
(Hint : Well-Ordering Property of natural numbers states that every nonempty subset of natural numbers has a *least element.*)

Exercise 2: Miscellaneous Mathematical Proofs *(2+3+3+1 Points)*

1. Let $S(n) = \sum_{i=1}^{n} i$ be the sum of the first n natural numbers and $C(n) = \sum_{i=1}^{n} i^3$ be the sum of the first n cubes. Use mathematical induction to prove the following interesting conclusion: $C(n) = S^2(n)$ for every n.

2. Let A, B, and C be subsets of U. Which of the following statements is true? Justify.
   - If $A \cap B = A \cap C$, then $B = C$.
   - If $A \cup B = A \cup C$, then $B = C$.
   - $\overline{A \cup B} = \overline{A} \cap \overline{B}$, where $\overline{A}$ is the compliment of A.

3. Let $A_1, A_2, ..., A_n$ be nonempty subsets of a Universal Set U, where n is any positive integer, and $n \geq 2$. Using the result of above exercise, i.e. $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$. Prove a generalized result

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$$

   using induction.

4. Let $A_1, A_2, ..., A_k$ be nonempty subsets of U, where k is any positive integer. Construct a nonempty subset $A \subseteq U$ such that $A \cap A_i \neq \phi$, for all $i \in \{1, 2, ..., k\}$.

Exercise 3: Graphs (Part 1) *(3+2 Points)*

A *simple graph* is a graph without self loops, i.e., every edge of the graph is an edge between two distinct nodes. The degree $d(v)$ of a node $v \in V$ in an undirected graph $G = (V, E)$ is the number of its neighbors, i.e, $d(v) = |\{u \in V \mid \{v, u\} \in E\}|$. Let $m \geq 0$ denote the number of edges in graph G.

1. Prove the handshaking lemma i.e. $\sum_{v \in V} d(v) = 2m$ via mathematical induction on m *for any simple graph* $G = (V, E)$.
2. Show that every simple graph with an odd number of nodes contains a node with even degree.

Exercise 4: Graphs (Part 2) *(2+4 Points)*

A graph $G = (V, E)$ is said to be *connected* if for every pair of vertices $u, v \in V$ such that $u \neq v$ there exists a path in G connecting u to v.

1. Prove that if G is connected, then for any two nonempty subsets V_1 and V_2 of V such that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$, there exists an edge joining a vertex in V_1 to a vertex in V_2.
2. Let G be a simple, connected graph and P be a path of the longest length ℓ in G. Show that if the two ends of P are adjacent, then $V = V(P)$, where $V(P)$ is the set of vertices of P.
   Hint: Try to argue by contradiction.