Albert-Ludwigs-Universität, Inst. für Informatik
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Algorithm Theory, Winter Term 2015/16
Problem Set 9

hand in (hard copy or electronically) by 10:15, Thursday January 07, 2016, tutorial session will be on January 11, 2016

Note: The points of this problem set are not required to be admitted to the exam. However, any points you achieve for this problem set count as bonus points towards getting 50% of the achievable points of all other problem sets.

Exercise 1: Maximum Matching (6 points)

We are given bipartite graph $B = (U \dot\cup V, E)$ on two disjoint node sets U and V; each edge connects a node in U and a node in V. In the following, we define a *2-claw* to be a set of three distinct nodes $\{u, x, y\}$ such that $u \in U$, $x, y \in V$ and there is an edge from u to both nodes x and y.
We consider the following maximization problem on the graph B: Find a largest possible set of vertex-disjoint set of 2-claws. In other words, we want to find a largest possible subset of edges such that every node in U is incident to either 0 or 2 of the edges and each node in V is incident to either 0 or 1 of the edges (i.e., each node is either part of one 2-claw or it is not part of any 2-claw at all).

(a) (3 Points) Show that picking vertex-disjoint 2-claws in a greedy manner (as long as there is a 2-claw which is vertex-disjoint to all previously picked 2-claws, we pick it) results in a set of 2-claws which is at least one-third as large as an optimal set of vertex-disjoint 2-claws.

(b) (3 Points) In order to solve the problem optimally, let us now assume that in the given bipartite graph B, each node in U has at most 3 neighbors in V. Give a polynomial-time algorithm which computes a maximum set of vertex-disjoint 2-claws. You can use algorithms from the lecture as a subroutine.

Hint: Try to reduce the problem to the maximum matching problem in general graphs.

Exercise 2: Triangles in Random Graphs (10 points)

Given a fixed vertex set $V = \{v_1, v_2, \ldots, v_n\}$ with n being an even number. Then the following (random) process defines the (undirected) random graph $G_p = (V, E_p)$:
For each vertex pair $\{v_i, v_j\}, i \neq j$ we independently decide with probability p whether the edge defined by this pair is part of the graph, i.e., whether $\{v_i, v_j\}$ is an element of the edge set E_p.
Furthermore we say that a subset $T = \{v_i, v_j, v_k\}$ of V of size 3 is a triangle of a graph, if all three edges $\{v_i, v_j\}$, $\{v_i, v_k\}$, $\{v_j, v_k\}$ are in the edge set of the graph.

(a) (1 Point) Let Z be the random variable that counts the number of edges in G_p. What kind of random variable is Z? What is the probability that Z has value k, for some k?

(b) (1 Point) Calculate m_T, the number of all triangles that could *possibly* occur in G_p.

(c) (2 Points) Let X denote the number of triangles in G_p. Calculate $\mathbb{E}[X]$.

The generation of the random graphs is now changed as follows. Before edges are determined each vertex is colored either red or green; we let K be the random variable that counts the number of red vertices. Between two red vertices there is an edge with probability p_{rr}, between two green vertices with probability p_{gg} and between vertices of different color with probability p_{rg} (edges are still picked independently).

(d) (3 Points) Assume first that with probability $\frac{1}{7}$ all vertices are red, with probability $\frac{2}{7}$ all vertices are green and with probability $\frac{4}{7}$ each vertex independently gets color red or green with probability 1/2 each. Also $p_{rr} = 1$, $p_{rg} = \frac{1}{\sqrt{3}}$ and $p_{gg} = 0$. Calculate $\mathbb{E}[X]$ under these conditions!

(e) (3 Points) Let us now assume that K is not known, but it is known that $K \sim \text{Uniform}[1, n]$, i.e., in the painting process first the number of red vertices is determined and then K vertices are being selected to be red. The edge probabilities are the same as in the question above. Consider a vertex $v \in V$. Conditioned on the event that v is red, what is the probability that v is *not part of any triangle*?