Theoretical Computer Science (Bridging Course)

Turing Machines

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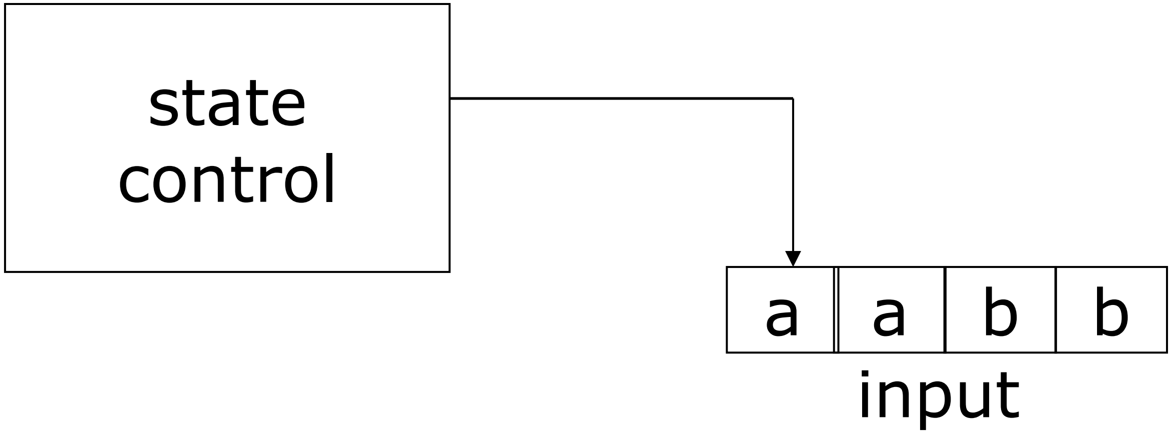
Topics Covered

- Turing machines
- Variants of Turing machines
  - Multi-tape
  - Non-deterministic
- Definition of algorithm
- The Church-Turing Thesis

Finite State Automata

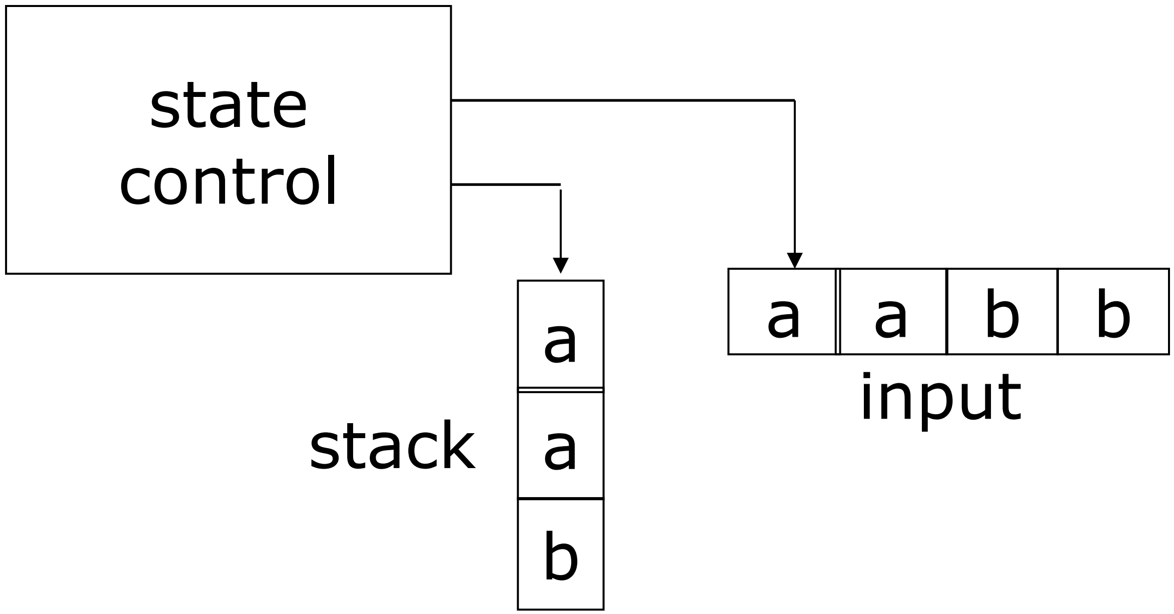
- Can be simplified as follow



- State control for states and transitions
- Tape to store the input string

Pushdown Automata

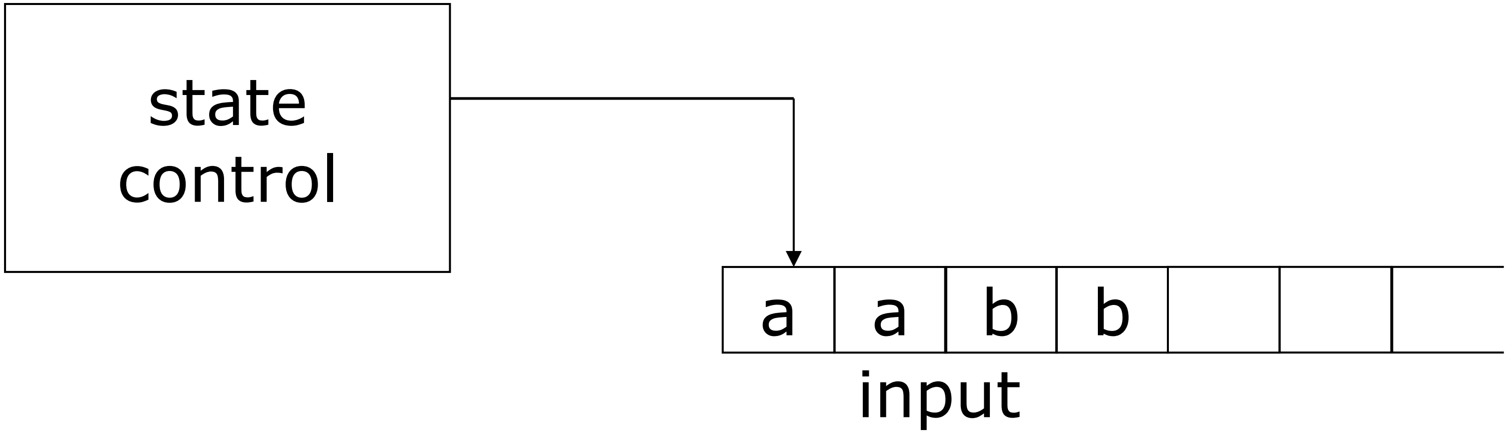
- Introduce a stack component



- Symbols can be read and written there

Turing Machine (TM)

- Introduce an infinite tape



- Symbols can be read and written there
- Move left and right on the tape
- Machine accepts, rejects, or loops

Turing Machine (TM)

- Let's design one for the language
$$F = \{w\#w \mid w \in \{0,1\}^*\}$$
- How will it work?

- Remember:
  - It has the string on the tape
  - It can go left and right
  - It can write symbols on the tape

Turing Machine (TM)

$$F = \{w\#w \mid w \in \{0,1\}^*\}$$

The machine does this:

- Scan to check there is only one #
- Zig-zag across # and read symbols
- If do not match reject
- If they match write the symbol x
- If all symbols left to # matche, accept

Turing Machine (TM)

$$F = \{w\#w \mid w \in \{0,1\}^*\}$$

$$w_1 \in F = \text{"011000\#011000"}$$

0	1	1	0	0	0	#	0	1	1	0	0	0	⊔	...
X	1	1	0	0	0	#	0	1	1	0	0	0	⊔	...
X	1	1	0	0	0	#	X	1	1	0	0	0	⊔	...
X	1	1	0	0	0	#	X	1	1	0	0	0	⊔	...
X	X	1	0	0	0	#	X	1	1	0	0	0	⊔	...
⋮														
X	X	X	X	X	X	#	X	X	X	X	X	X	⊔	...

Formal Definition of a TM

A Turing machine is a 7-tuple

$$(Q, \Sigma, \Gamma, \delta, q_o, q_{accept}, q_{reject})$$

- Q is the set of states
- Σ is the input alphabet, without $\sqcup$
- Γ is the tape alphabet and $\sqcup \in \Gamma, \Sigma \subseteq \Gamma$
- $\delta: Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$ is the transition function
- $q_0 \in Q$ is the initial state
- $q_{accept}, q_{reject} \in Q$ are the final states

TM Configurations

- Describe the state of the machine
- Written as $C = uq_iv$ where:
  - q_i is the current state of the machine
  - uv is the content of the tape
  - The head stays at the first symbol of v

TM Transitions

- A configuration C_1 yields C_2 if the machine can go from C_1 to C_2 in 1 step
- uaq_ibv yields uq_jacv if $\delta(q_i, b) = \left(q_j, c, L\right)$
- uaq_ibv yields $uacq_jv$ if $\delta(q_i, b) = \left(q_j, c, R\right)$
- Note: cannot go over the left border!

TM Acceptance

- The machine starts at $q_0 w$
- The machine accepts at q_{accept}
- The machine rejects at q_{reject}

- An input is accepted if there is $C_1, \dots, C_k$
  - The machine starts at C_1
  - Each C_i yields C_{i+1}
  - C_k is an accepting state

Computations and Deciders

- Three possible outcomes:
  - It ends in an accept state
  - It ends in a reject state
  - It does not end (loops forever)

- Accept and reject are halting states
- Loops are not halting
- A **Decider** halts on every input

TMs and Languages

- The strings a TM M accepts define the language of M, $\mathrm{L}(M)$
- A language is Turing recognizable (recursively enumerable) if some TM recognizes it
- A language is Turing decidable (recursive) if some TM decides it

TM Example

TM M_2 recognizes the language consisting of all strings of zeros with their length being a power of 2. In other words, it decides the language

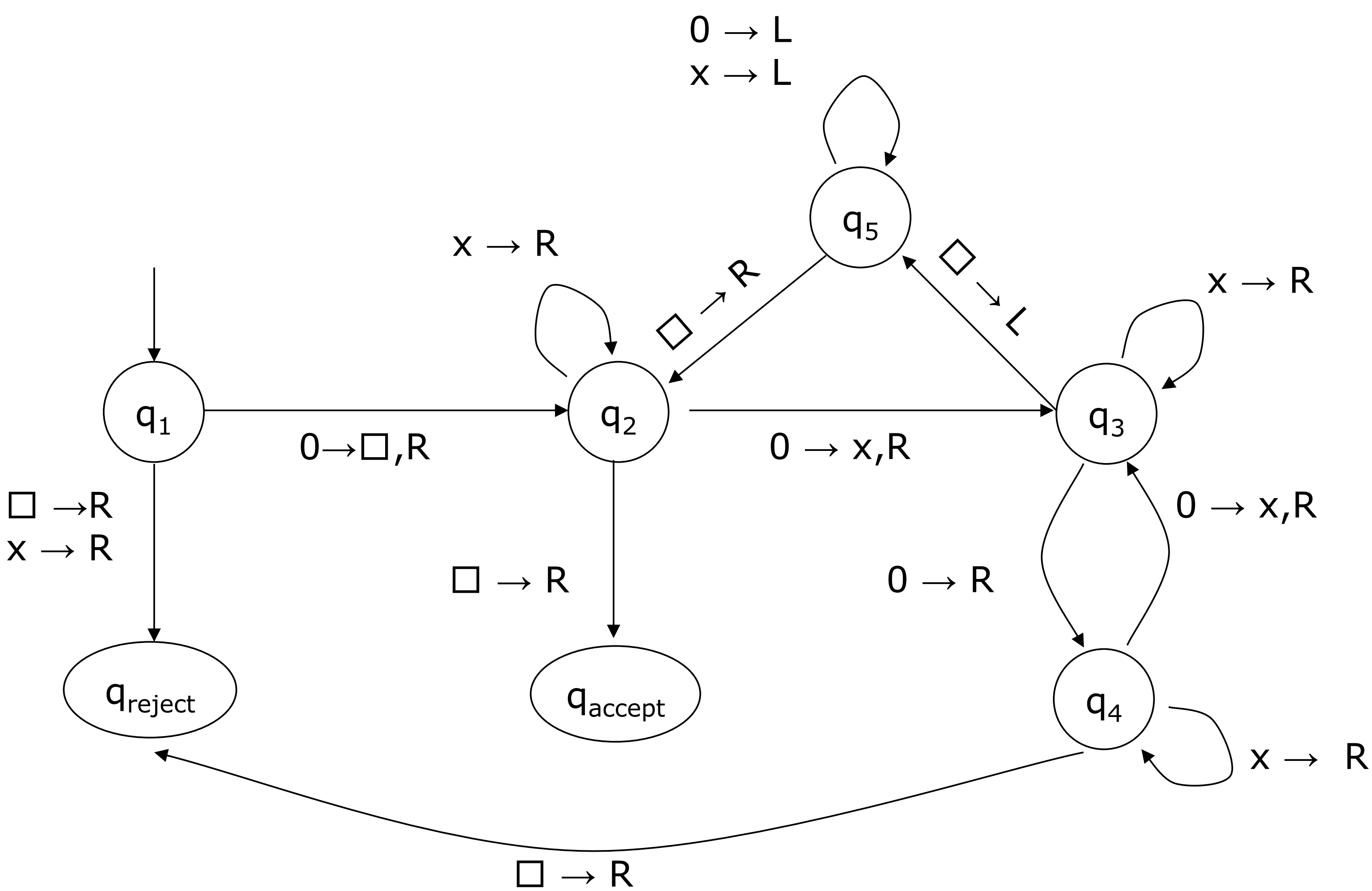
$$A = \left\{0^{2^n} \,\middle|\, n \geq 0\right\}.$$

TM Example

$$A = \{0^{2^n} \mid n \geq 0\}$$

1. Sweep left to right accross the tape, crossing off every other 0
2. If the tape has a single 0, *accept*
3. If the tape has more than one 0 and the number of 0s is odd, *reject*
4. Return the head to the left
5. Go to stage 1

TM Example



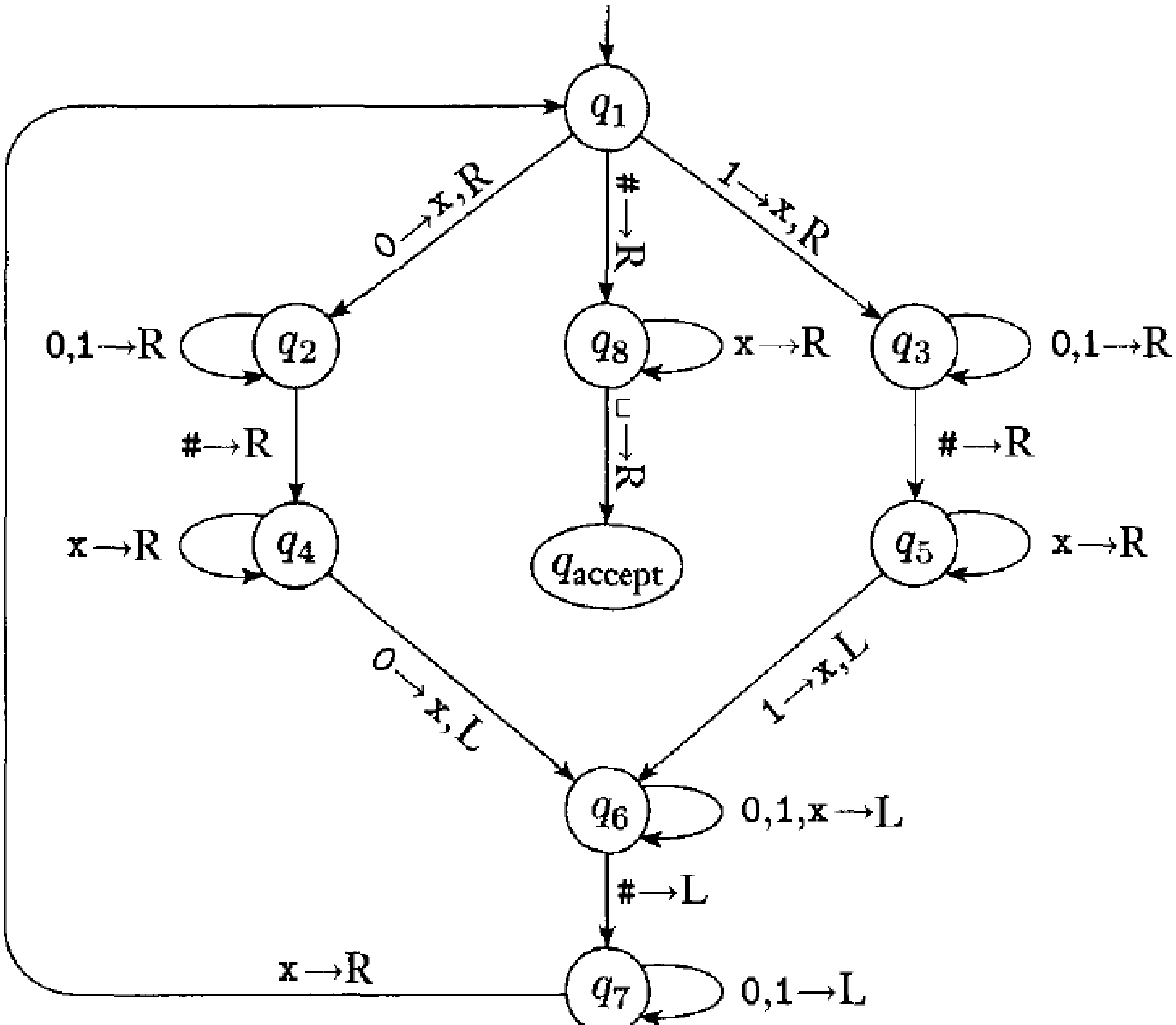
0 → L
x → L
q5
x → R
◇ → R
◇ → L
x → R
q1
q2
q3
0→□,R
0 → x,R
□ →R
x → R
0 → x,R
□ → R
0 → R
qreject
qaccept
q4
x → R
□ → R

Another TM Example

$$F = \{w\#w \mid w \in \{0,1\}^*\}$$

1. Check for #, if not *reject*
2. Zig-zag across and cross off same symbols. If not same, *reject*
3. If all left of # are crossed, check for non crossed symbols on the right side
4. If none, *accept,* otherwise *reject*

Another TM Example



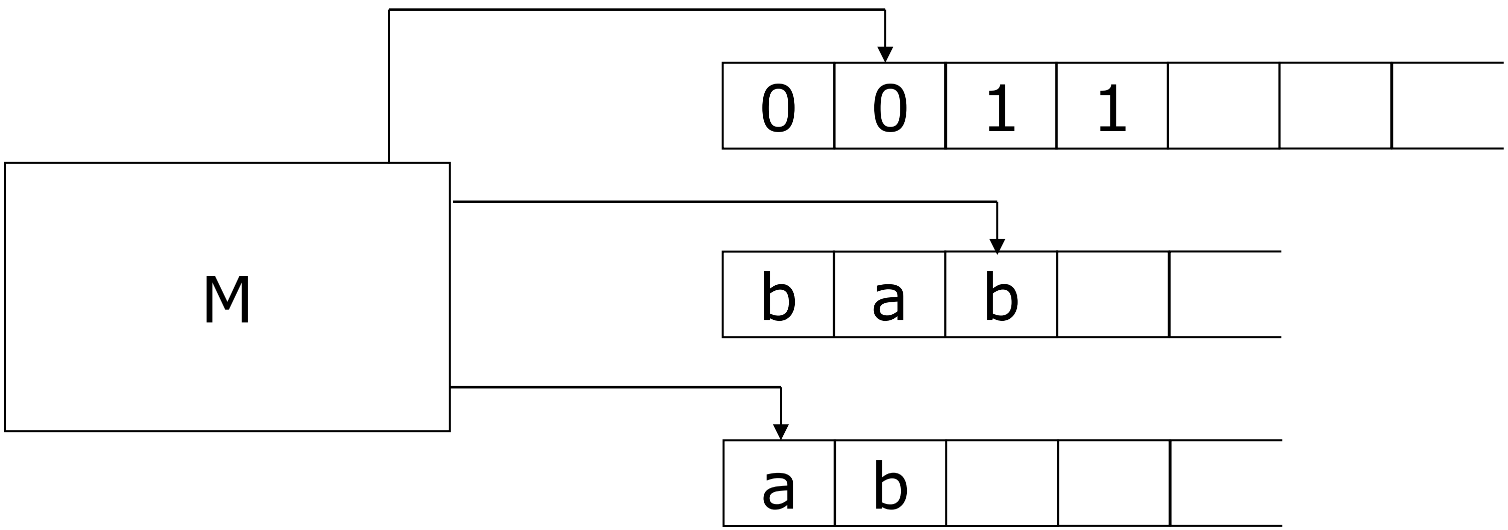
Variants of Turing Machines

- Mostly equivalent to the original

- Example: consider movements as {L,R,S}, where S means stay still
- Equivalent to original, represent S as two transitions: first R, then L or vice versa

Multi-Tape Turing Machine

- Include multiple tapes and heads



- Input on first tape, the others blank
- Transitions $\delta: Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R, S\}^k$

Equivalence Result

Theorem 3.13:

Every multitape Turing machine has an equivalent single-tape Turing machine.

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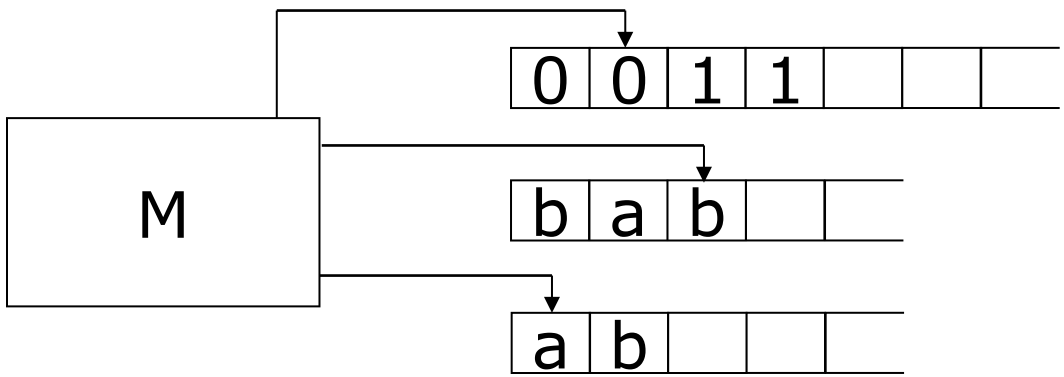

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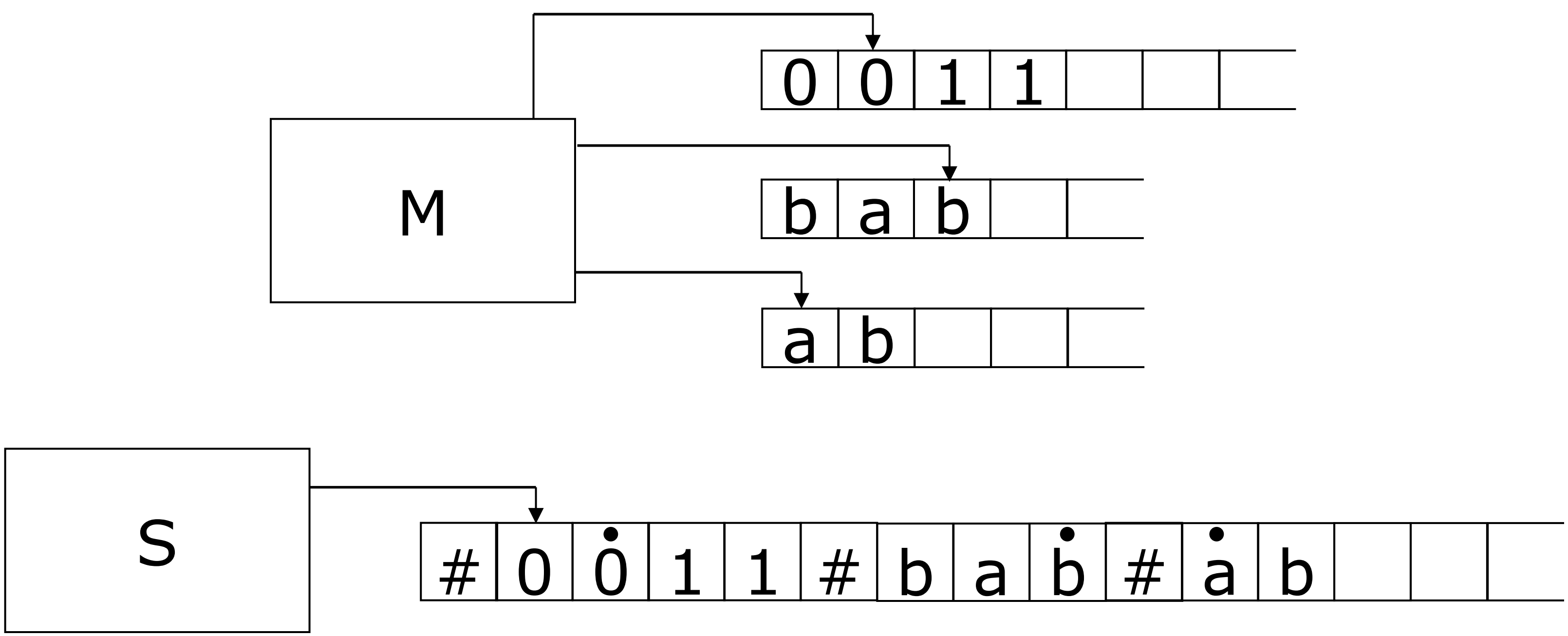


Proof of Theorem 3.13

- Consider a input $w_1 w_2 \dots w_k$
- Add dotted symbols for the head
- Put all the input on the single tape

$$\#\dot{w}_1 w_2 \dots w_k \# \dot{\sqcup} \# \dot{\sqcup} \# \dots \#$$

- Simulate a single move
  - Scan from first # to last to get the heads
  - Re-run to update the tape
- If head symbols go to the right # write a blank and shift the tape content

Equivalence Result

Corollary 3.15:

A language is Turing-recognizable if and only if some multi-tape Turing machine recognizes it

Proof:

Forward: an ordinary machine is a special case of a multi-tape

Backward: see Theorem 3.13

Intermezzo: Programming

"Brainfuck": language *simulating* a TM

Character	Meaning
>	increment the data pointer (to point to the next cell to the right). R
<	decrement the data pointer (to point to the next cell to the left). L
+	increment (increase by one) the byte at the data pointer.
-	decrement (decrease by one) the byte at the data pointer.
.	output a character, the ASCII value of which being the byte at the data pointer.
,	accept one byte of input, storing its value in the byte at the data pointer.
[	if the byte at the data pointer is zero, then instead of moving the instruction pointer forward to the next command, jump it forward to the command after the matching] command.
]	if the byte at the data pointer is nonzero, then instead of moving the instruction pointer forward to the next command, jump it back to the command after the matching [command*.

(http://en.wikipedia.org/wiki/Brainfuck)

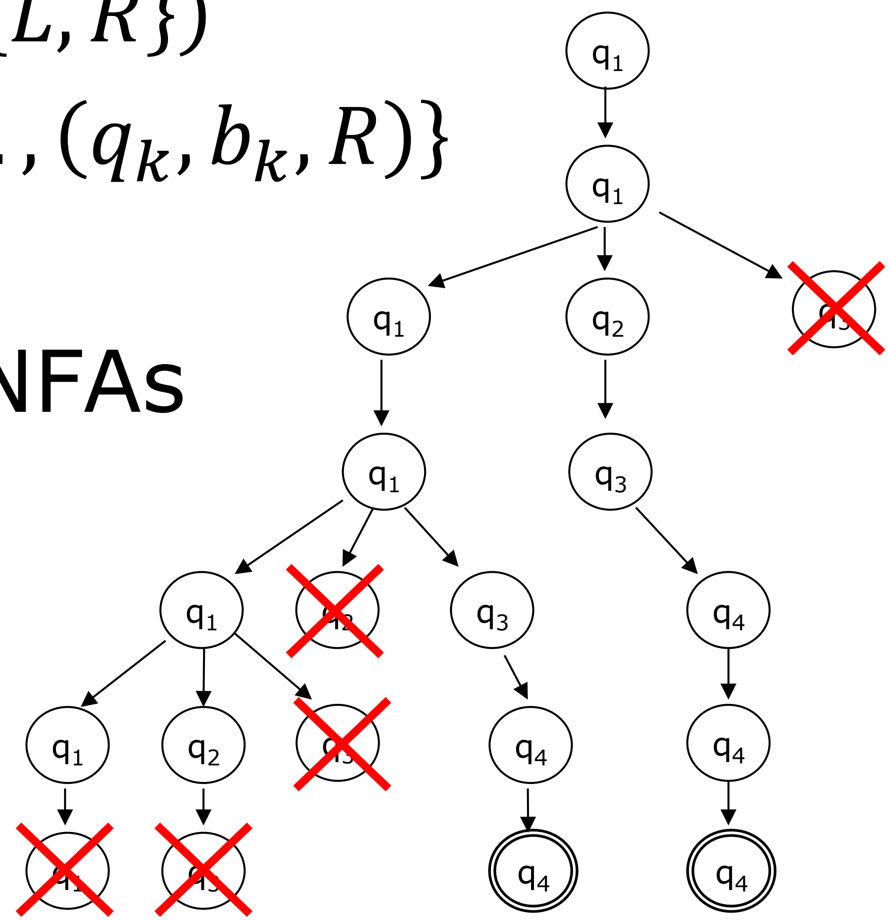
Non Deterministic TMs (NTMs)

- Transition function changed into

$$\delta: Q \times \Gamma \to P(Q \times \Gamma \times \{L, R\})$$

$$\delta(q, a) = \{(q_1, b_1, L), \dots, (q_k, b_k, R)\}$$

- Same idea as for NFAs



Equivalence of NTMs and TMs

<u>Theorem 3.16:</u>

Every nondeterministic Turing machine has an equivalent deterministic Turing machine.

Idea:

- Three tapes: input, simulation, index
- Simulator to perform computation
- Index to trace the path in the tree

Equivalence of NTMs and TMs

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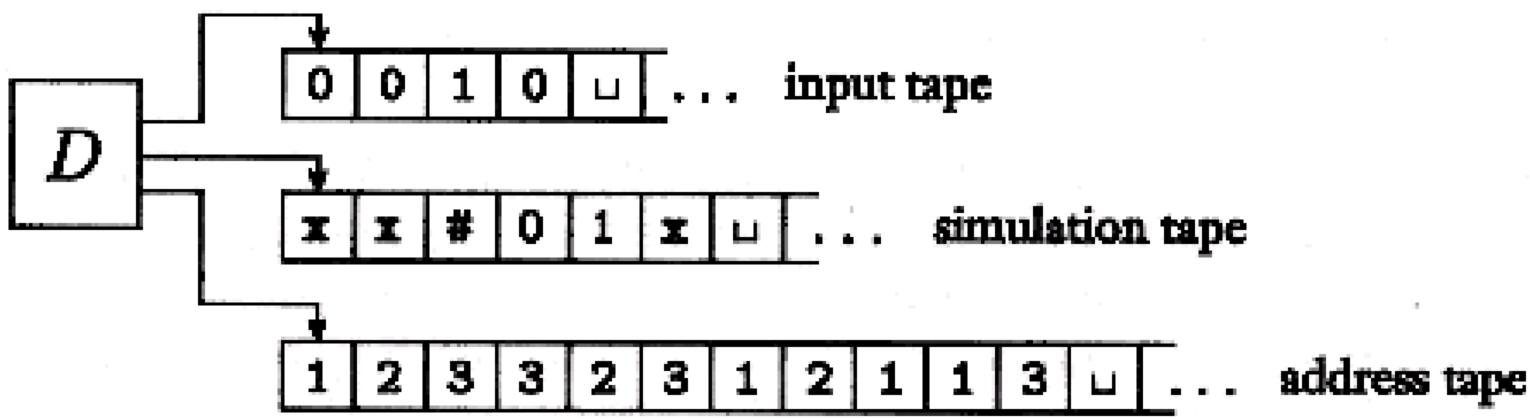


Proof of Theorem 3.16

1. Copy the input from tape 1 to 2
2. Use tape 2 to simulate N on one branch of computation
    a. Consult tape 3 to get the transition
    b. Abort if empty symbol, invalid or reject
    c. Accept if accept state
3. Replace the string on 3 with the lexicographically next one
4. Repeat from 1.

NTMs and Languages

Corollary 3.18:

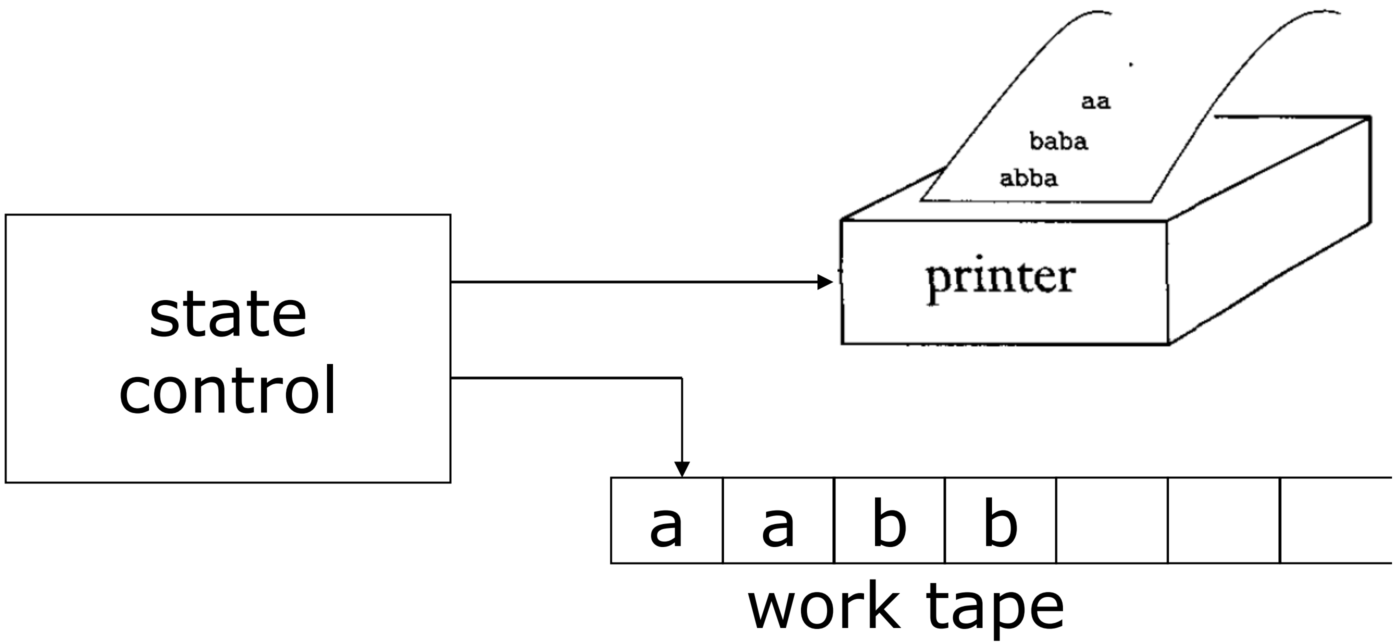
A language is Turing-recognizable if and only if some nondeterministic Turing machine recognizes it.

Corollary 3.19:

A language is decidable if and only if some nondeterministic Turing machine decides it.

Enumerators

- Recursively enumerable languages
- Recognized by TMs
- Alternative model: Enumerator



Enumerators

- *Enumerate* the strings
- Start with empty tape
- Output tape (printer)
- Print strings instead of accepting them

- Printing in any order
- Strings might be duplicated

Equivalence Result

Theorem 3.21:

A language is Turing-recognizable if and only if some enumerators enumerate it.

Proof:

Forward: e have an enumerator E.

We can build a machine T that

1. Run E and compare every string
2. If it appears, accept

Equivalence Result

Backward: We have a machine T.

We can build an enumerator E as this:

1. Ignore the input
2. For each i = 1,2,...
   1. Run T for i steps on each input in Σ^*
   2. If any computation accepts, print it.

E eventually prints all string T accepts

Other Variants of TMs

- Many other variants of TMs exist
- All equivalent in power under reasonable assumptions
- *Turing complete* languages
- The class of algorithms described identical for all these languages.
- For a given task, one type of language may be more elegant or *simple*.

Definition of Algorithm

- Precise definition only in 20^{th} century
- Informal idea was already present
- Collection of instructions for a task
- Formal definition needed to be found

Anecdote: David Hilbert

- Famous mathematician
- Int. congress of Maths in 1900
- Formulated 23 math problems

- The 10th problem said:
  - Devise an *algorithm* to test whether a polynomial has an integral root
  - Algorithm = "a process according to which it can be determined by a finite number of operations"

Anecdote: David Hilbert

- Mathematicians believed it existed
- We know it is not possible
- A formal definition of algorithm was needed to prove it
- Alonso Church : λ-calculus
- Alan Turing: Turing machines
- Church—Turing Thesis:
  - Intuitive algorithm = TM algorithm

Formal Definition of Algorithm

- Let’s rephrase Hilbert problem
- Consider the set

  $D = \{p \mid p \text{ is a polynomial with integer root}\}$
- Hilbert problem asks if D is decidable
- Unfortunately it is not
- Fortunately is Turing recognizable

Formal Definition of Algorithm

- Consider a simpler problem

$$D_1 = \{p \mid p \text{ is a poly. over } x \text{ with integer root}\}$$

- Build a TM that recognizes it

1. Input is a polynomial over x
2. Evaluate p with x=0,1,-1,2,-2,...
3. If polynomial evaluates to 0, accept

Formal Definition of Algorithm

- Describe an algorithm equals to describe a Turing machine
- Three possibilities:
  - Formal description (low level)
  - Implementation description (mid level)
  - *English* description (high level)

- We will describe machines in high level

Turing Machine Description

- Input is always a string
- Objects represented as strings
- Encoding is irrelevant (equivalence)
- TM Algorithm will be high level
- First line describe the input
- Indentations describe blocks

Example description

$A = \{\langle G \rangle \mid G \text{ is a connected undirected graph}\}$

Remember the definition of connected?

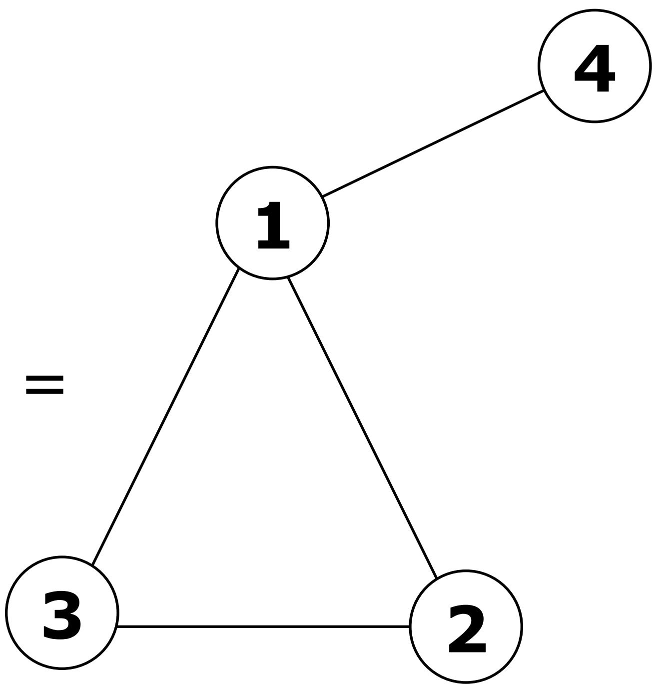
Example description

$A = \{\langle G \rangle |\ G \text{ is a connected undirected graph}\}$

Remember the definition of connected?

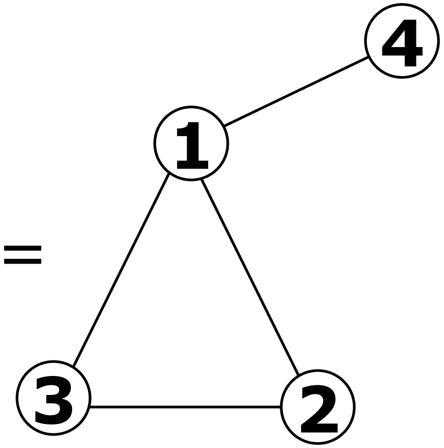
Every node is reachable from every one

G =



Example description

G =



<G> = (1,2,3,4) ((1,2),(2,3),(3,1),(1,4))

M = „On input <G>, the encoding of a graph G:

1. Select the first node of G and mark it.
2. Repeat the following stage until no new nodes are marked.
    1. For each node in G, mark it if it is attached by an edge to a node that is already marked.
3. Scan all the nodes of G to determine whether they all are marked.
   If yes, accept; otherwise reject.“

Summary

- Turing machines
- Variants of Turing machines
  - Multi-tape
  - Non-deterministic
- The definition of algorithm
  - The Church-Turing Thesis