Algorithm Theory

Chapter 2
Greedy Algorithms

Part I:
Interval Scheduling & Partitioning

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Greedy Algorithms

- No clear definition, but essentially:

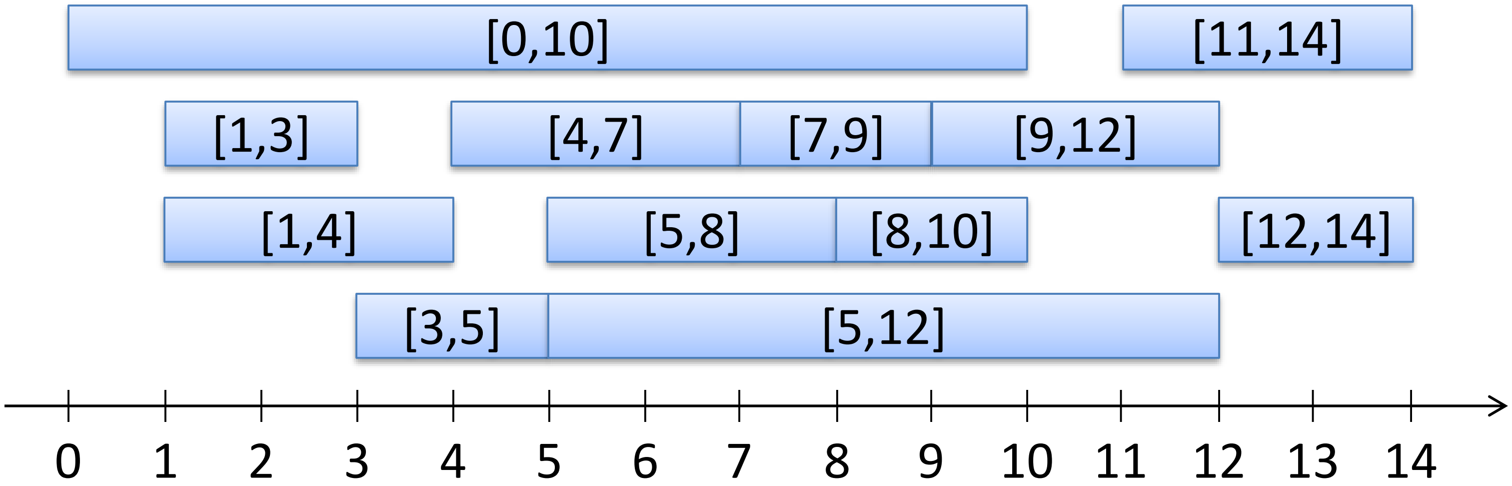
In each step make the choice that looks best at the moment!

- Depending on problem, greedy algorithms can give
  - Optimal solutions
  - Close to optimal solutions
  - No (reasonable) solutions at all
- If it works, very interesting approach!
  - And we might even learn something about the structure of the problem

Goal: Improve understanding where it works (mostly by examples)

Interval Scheduling

- **Given:** Set of intervals, e.g.
  [0,10],[1,3],[1,4],[3,5],[4,7],[5,8],[5,12],[7,9],[9,12],[8,10],[11,14],[12,14]



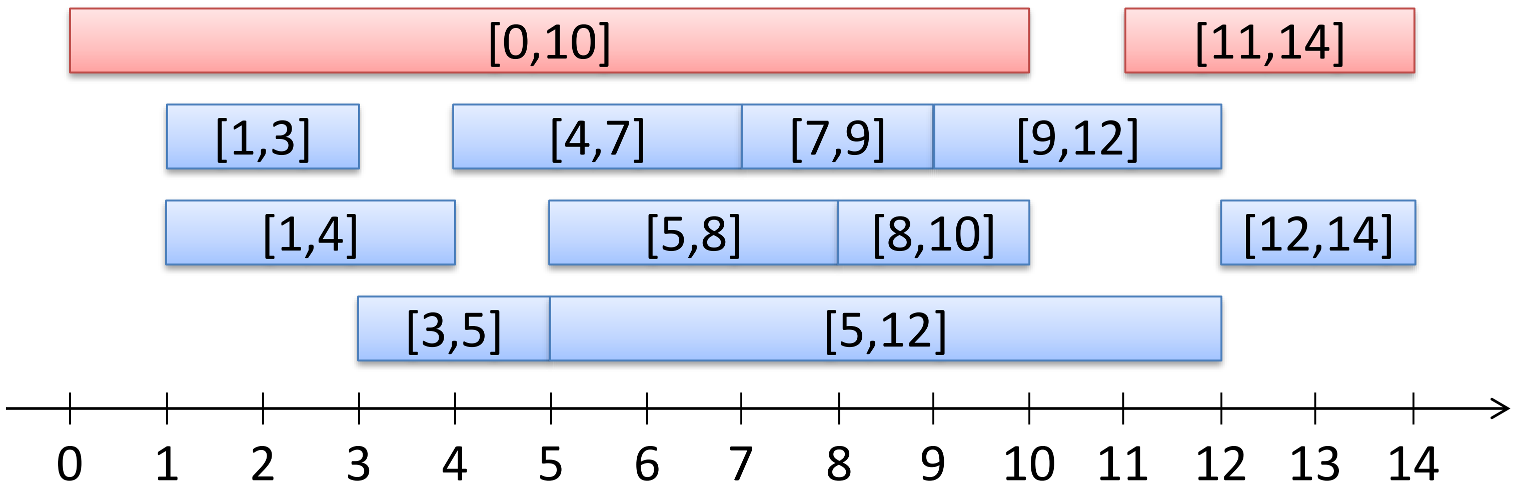
- **Goal:** Select largest possible non-overlapping set of intervals
  - For simplicity: overlap at boundary ok
    (i.e., [4,7] and [7,9] are non-overlapping)

- **Example:** Intervals are room requests; satisfy as many as possible

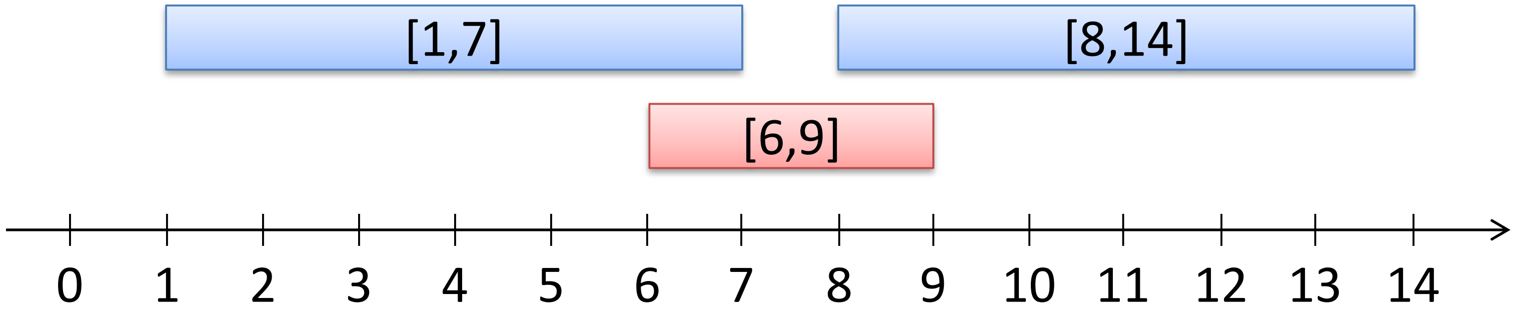
Greedy Algorithms

- Several possibilities…

Choose first available interval:

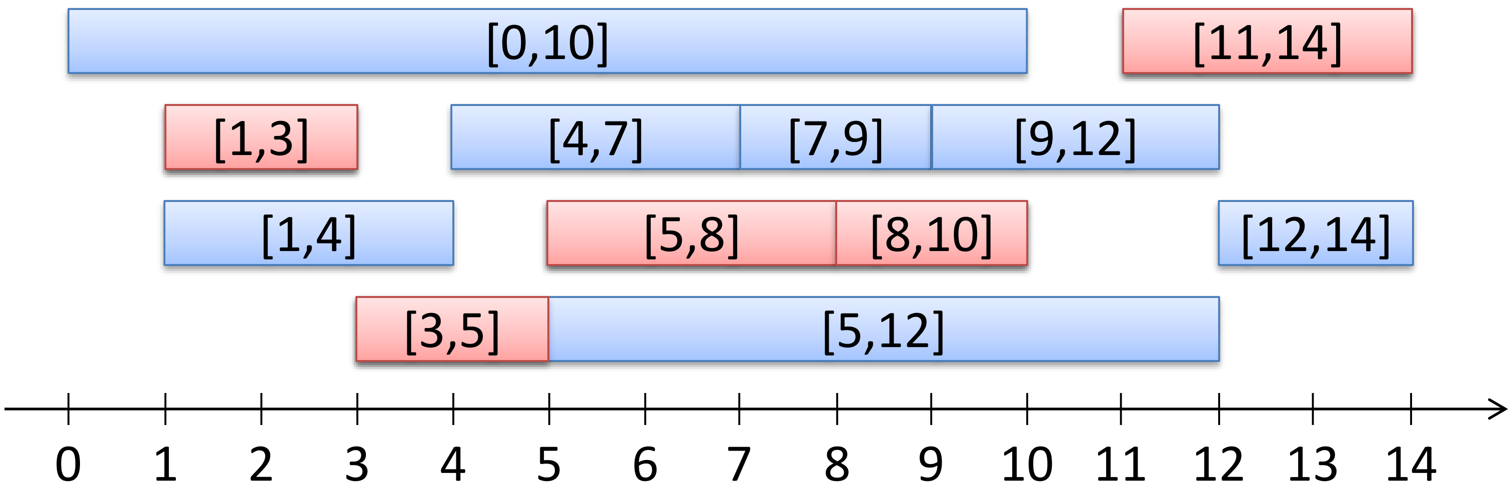


Choose shortest available interval:



Greedy Algorithms

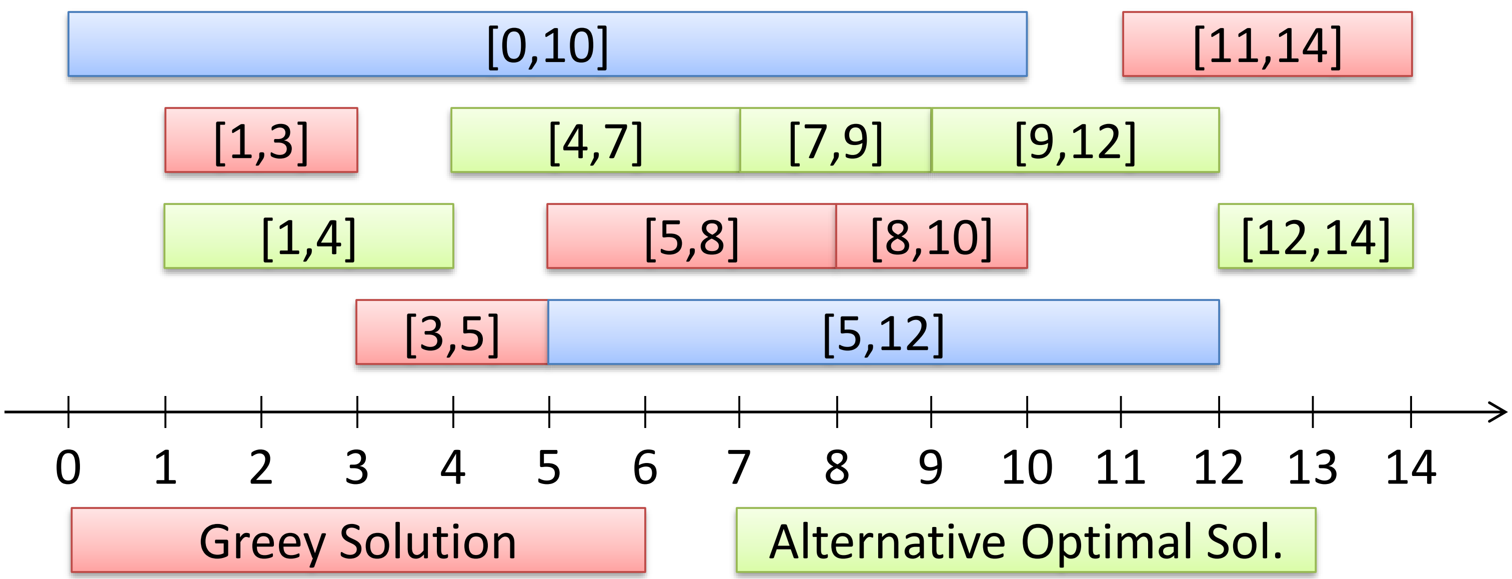
Choose available request with earliest finishing time:



R := set of all requests; S := empty set;
while R is not empty **do**
    choose $r \in R$ with smallest finishing time
    add r to S
    delete all requests from R that are not compatible with r
end   // S is the solution

Earliest Finishing Time is Optimal

- Let O be the set of intervals of an optimal solution
- Can we show that $S = O$?
  - No…



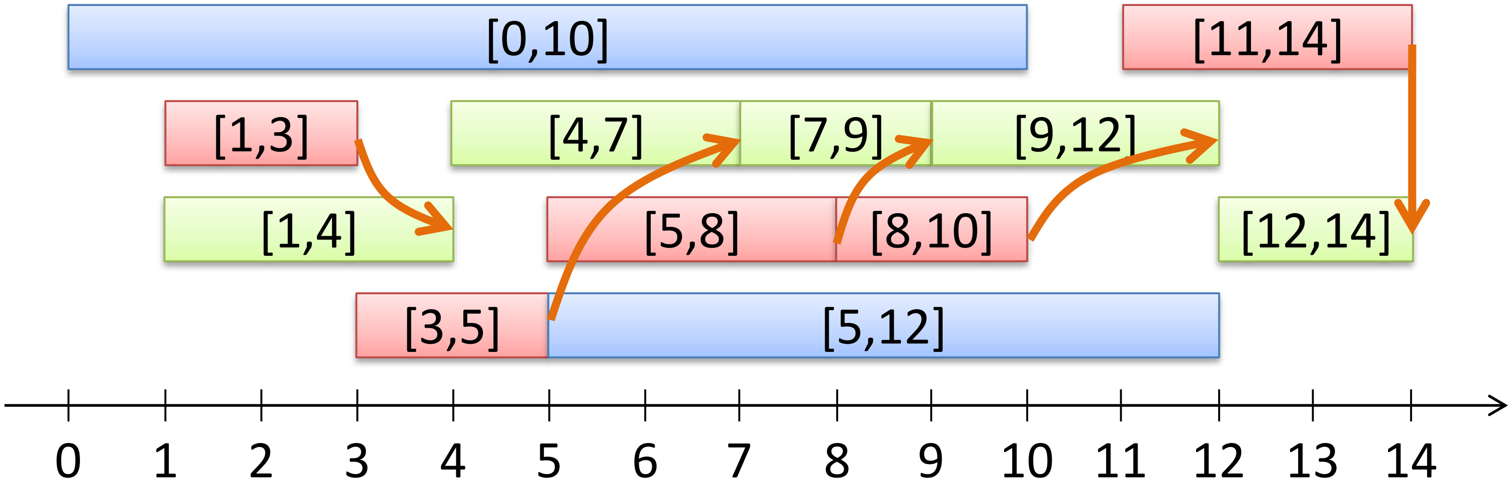
- Show that $|S| = |O|$.

Or alternatively: $|S| \geq |O|$ for any other solution O.

Greedy Stays Ahead

- Greedy solution S:
  $$[a_1, b_1], [a_2, b_2], \dots, [a_{|S|}, b_{|S|}], \qquad \text{where } b_i \le a_{i+1}$$
- Some optimal solution O:
  $$[a_1^*, b_1^*], [a_2^*, b_2^*], \dots, [a_{|O|}^*, b_{|O|}^*], \qquad \text{where } b_i^* \le a_{i+1}^*$$
- Define $b_i := \infty$ for $i > |S|$ and $b_i^* := \infty$ for $i > |O|$

Claim: $\forall i \ge 1,\ b_i \le b_i^*$ $\implies |S| \ge |O|$ because $b_{|O|} \le b_{|O|}^* < \infty$



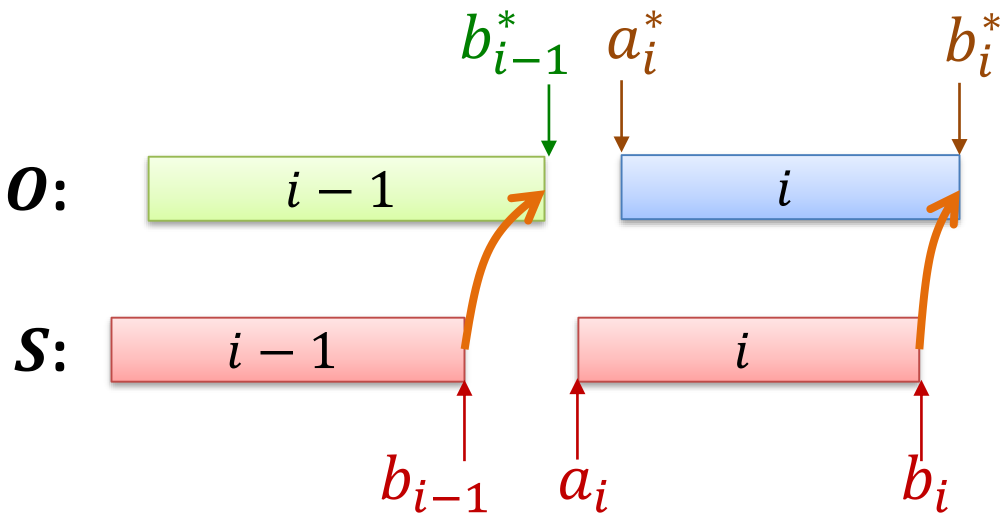
Greedy Stays Ahead

Claim: For all $i \geq 1$, $b_i \leq b_i^*$

Proof (by induction on i):

Base case $i = 1$: $\boldsymbol{b_1 \leq b_1^*}$ ✓

Step $i - 1 \rightarrow i$: Induction Hypothesis: $\boldsymbol{b_{i-1} \leq b_{i-1}^*}$



Need to show that $\boldsymbol{b_i \leq b_i^*}$:

Blue interval is available to greedy algorithm because

$$b_{i-1} \leq b_{i-1}^* \leq a_i^*$$

Greedy would prefer blue interval if $b_i^* < b_i$. ✓

Corollary: Earliest finishing time algorithm is optimal.

Weighted Interval Scheduling

Weighted version of the problem:

- Each interval has a weight
- Goal: Non-overlapping set with maximum total weight

Earliest finishing time greedy algorithm fails:

- Algorithm needs to look at weights
- Else, the selected sets could be the ones with smallest weight…

No simple greedy algorithm:

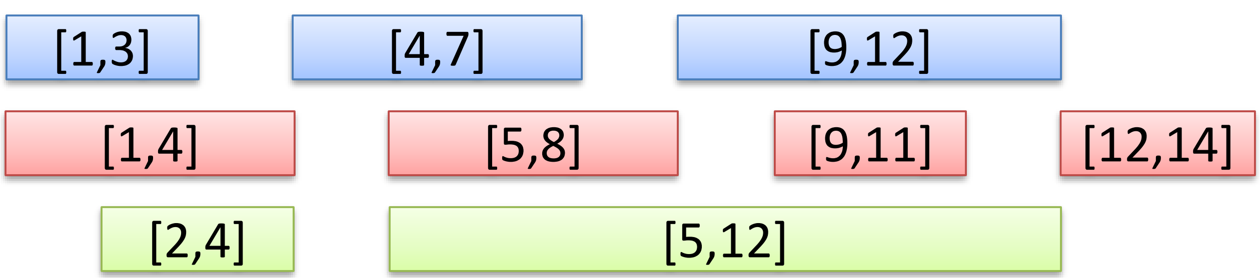
- We will see an algorithm using another design technique later.

Interval Partitioning

- **Schedule all intervals**: Partition intervals into as few as possible non-overlapping sets of intervals
  – Assign intervals to different resources, where each resource needs to get a non-overlapping set

- Example:
  – Intervals are requests to use some room during this time
  – Assign all requests to some room such that there are no conflicts
  – Use as few rooms as possible

- Assignment to 3 resources:

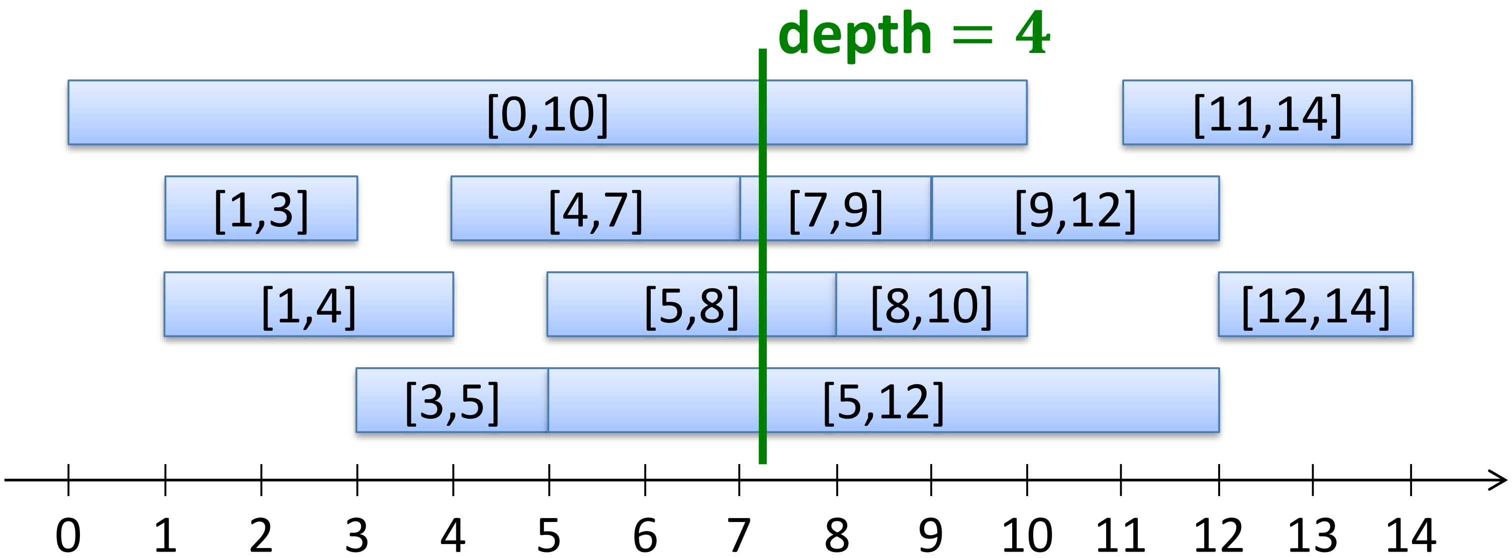


Depth

Depth of a set of intervals:

- Maximum number passing over a single point in time
  - Because we allow intervals to overlap at the boundaries, “passing” means in the inside of the interval.

- Depth of initial example is 4 (e.g., [0,10],[4,7],[5,8],[5,12]):



Lemma: Number of resources needed $\geq$ depth

- Follows directly from definition of depth.

Greedy Algorithm

Can we achieve a partition into "depth" non-overlapping sets?

- Would mean that the only obstacles to partitioning are local…

Algorithm:

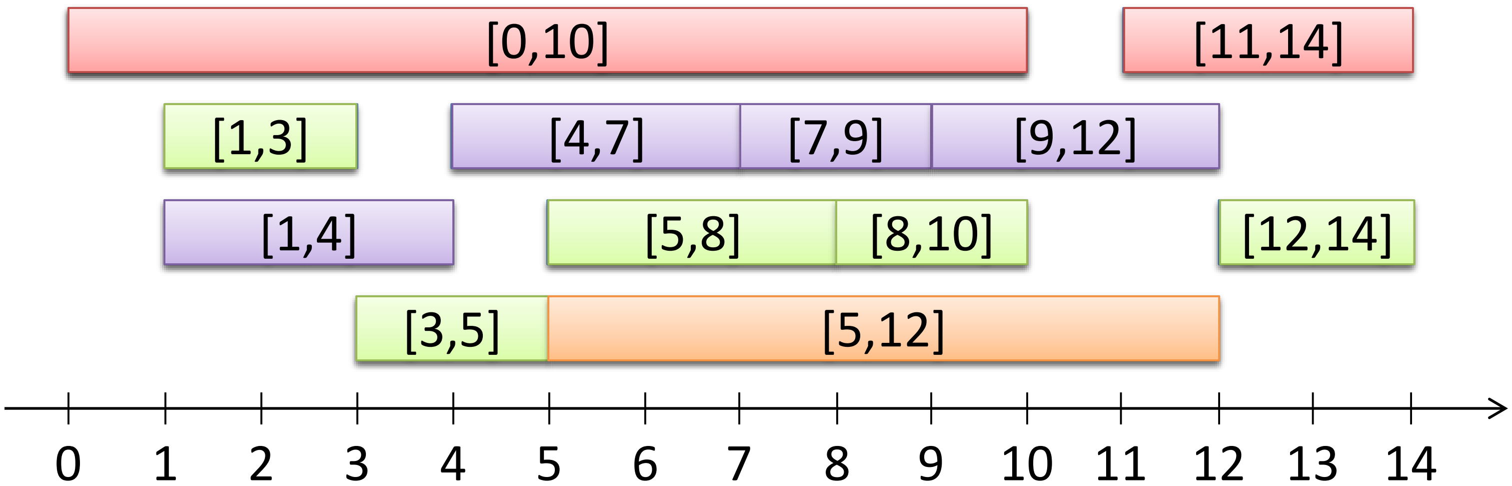
- Assign labels 1, … to the intervals; same label → non-overlapping

1. sort intervals by starting time: $I_1, I_2, \dots, I_n$
2. **for** $i = 1$ **to** n **do**
3. assign smallest possible label to I_i
   (possible label: different from conflicting intervals $I_j, j < i$)
4. **end**

Interval Partitioning Algorithm

Example:

- Labels: 



- Number of labels = depth = 4

Interval Partitioning: Analysis

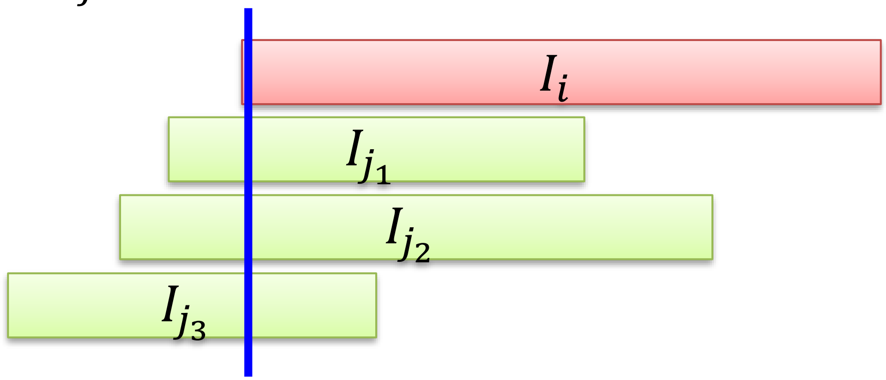
Theorem:

a) Let d be the depth of the given set of intervals. The algorithm assigns a label from $1, \dots, d$ to each interval.

b) Sets with the same label are non-overlapping

Proof:

- b) holds by construction
- For a):
  - All intervals $I_j, j < i$ overlapping with I_i, overlap at the beginning of I_i



  - At most $d - 1$ such intervals → some label in $\{1, \dots, d\}$ is available.