Algorithm Theory

Chapter 2
Greedy Algorithms

Part II:
Traveling Salesperson Problem (TSP)

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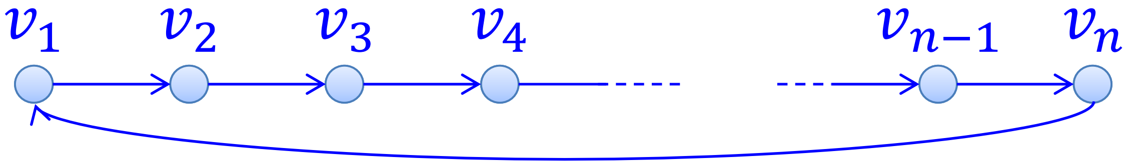
Traveling Salesperson Problem (TSP)

Input:

- Set V of n nodes (points, cities, locations, sites)
- Distance function $d: V \times V \to \mathbb{R}$, i.e., $d(u, v)$: dist. from u to v
- Distances usually symmetric, asymm. distances → asymm. TSP

Solution:

- Ordering/permutation $v_1, v_2, \dots, v_n$ of nodes

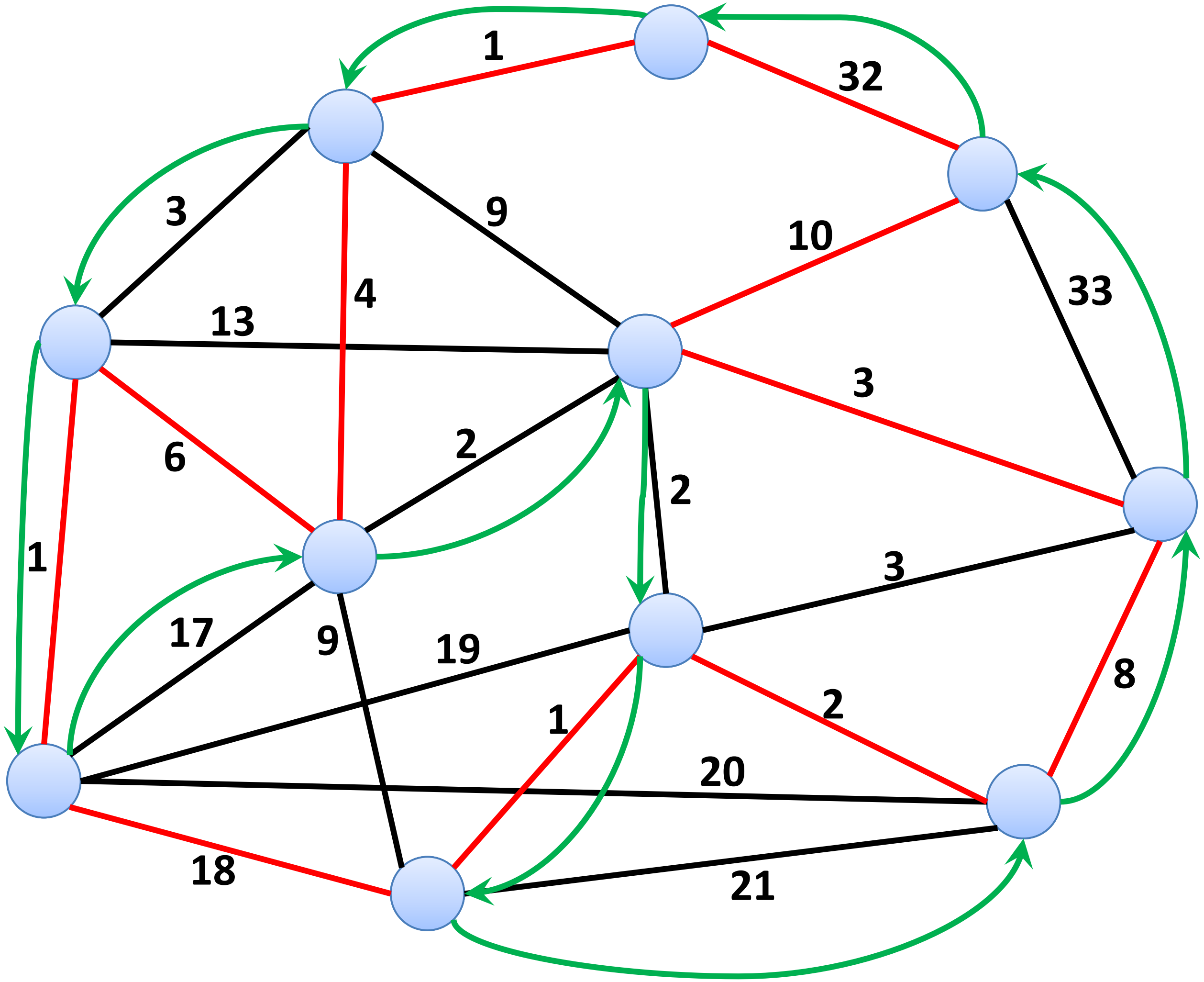


- Length of TSP path: $\sum_{i=1}^{n-1} d(v_i, v_{i+1})$
- Length of TSP tour: $d(v_n, v_1) + \sum_{i=1}^{n-1} d(v_i, v_{i+1})$

Goal:

- Minimize length of TSP path or TSP tour

Example



Greedy Algorithm

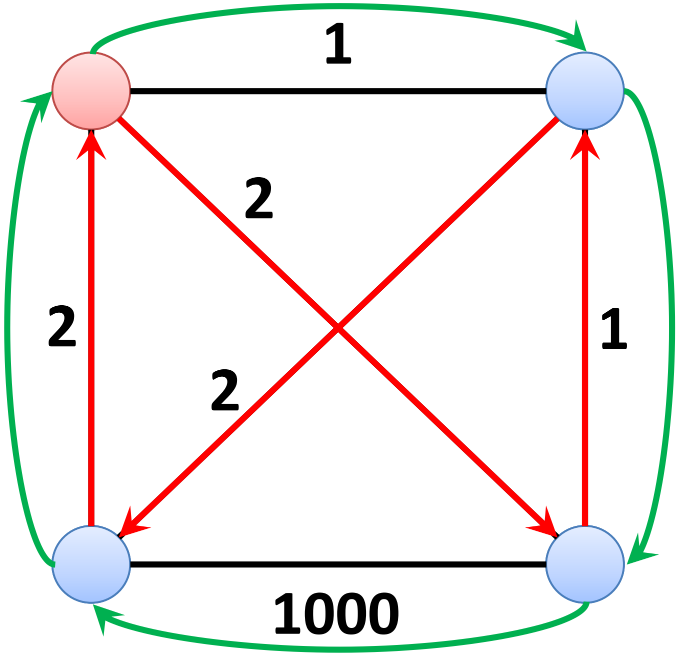
Length: 121

Optimal Tour

Length: 86

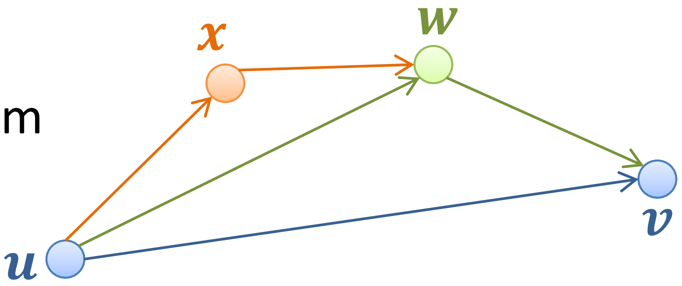
Nearest Neighbor (Greedy)

- Nearest neighbor can be arbitrarily bad, even for TSP paths



TSP Variants

- Asymmetric TSP
  - arbitrary non-negative distance function
  - most general, nearest neighbor arbitrarily bad
  - NP-hard to get within any bound of optimum
- Symmetric TSP
  - arbitrary non-negative symmetric distance function
  - nearest neighbor arbitrarily bad
  - NP-hard to get within any bound of optimum

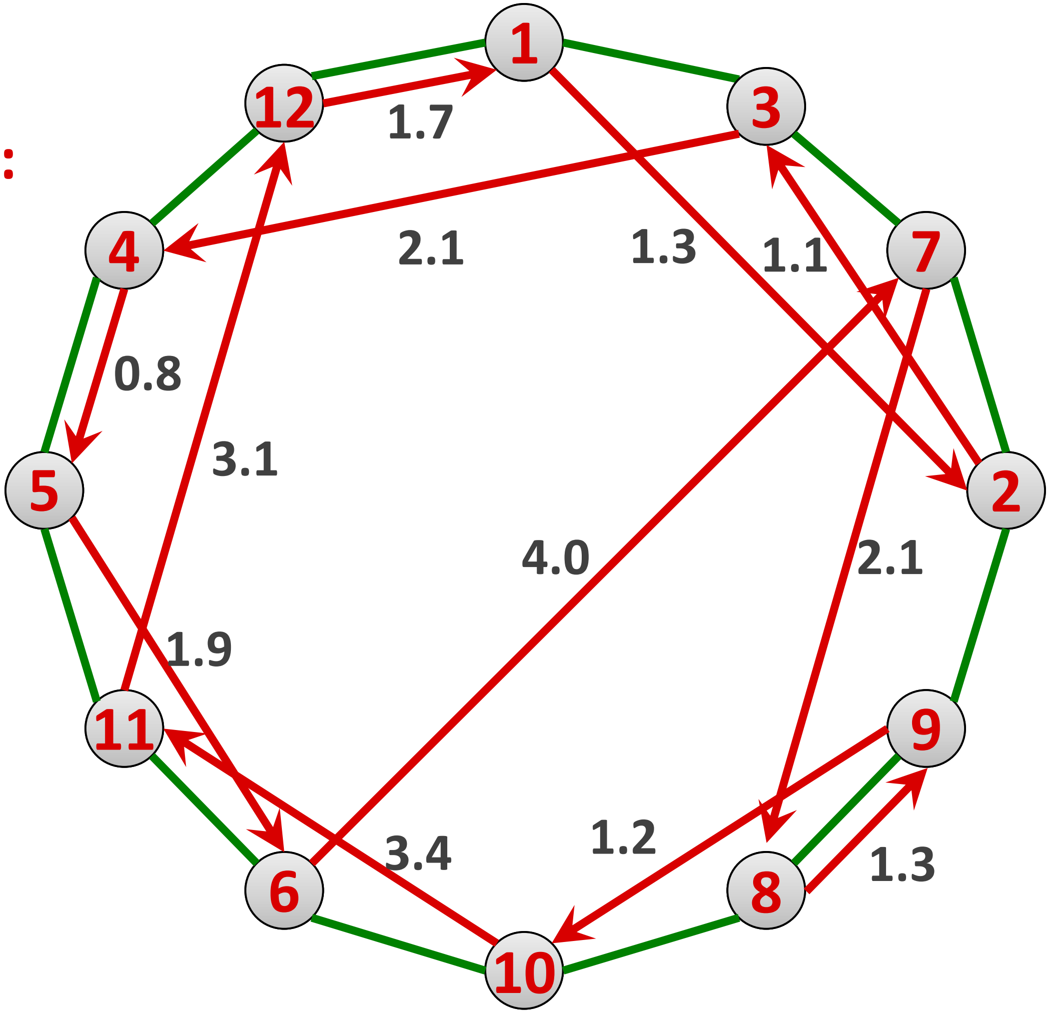


- Metric TSP
  - distance function defines metric space: symmetric, non-negative, triangle inequality: $d(u,v) \leq d(u,w) + d(w,v)$
  - possible to get close to optimum (we will later see factor ${}^{3}/_{2}$)
  - what about the nearest neighbor algorithm?

Metric TSP, Nearest Neighbor

Optimal TSP tour:

Nearest-Neighbor TSP tour:



Metric TSP, Nearest Neighbor

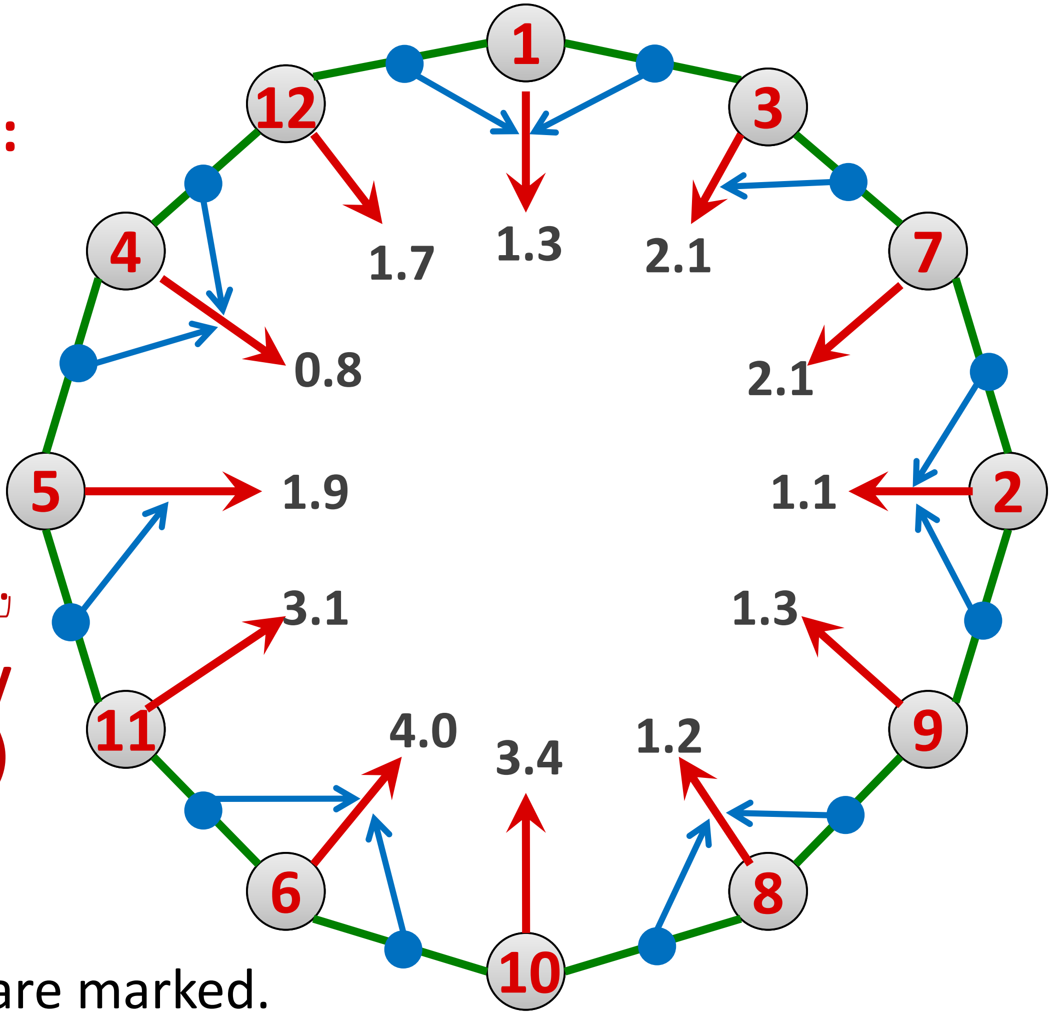
Optimal TSP tour:

Nearest-Neighbor TSP tour:

cost = 24

marked red edges :
some arrow to it

$\underbrace{\text{green edges}}_{\textbf{OPT}} \geq \underbrace{\text{marked red ed.}}_{\textbf{part of greedy solution (NN)}}$



#marked red edges:

At least half of the red edges are marked.

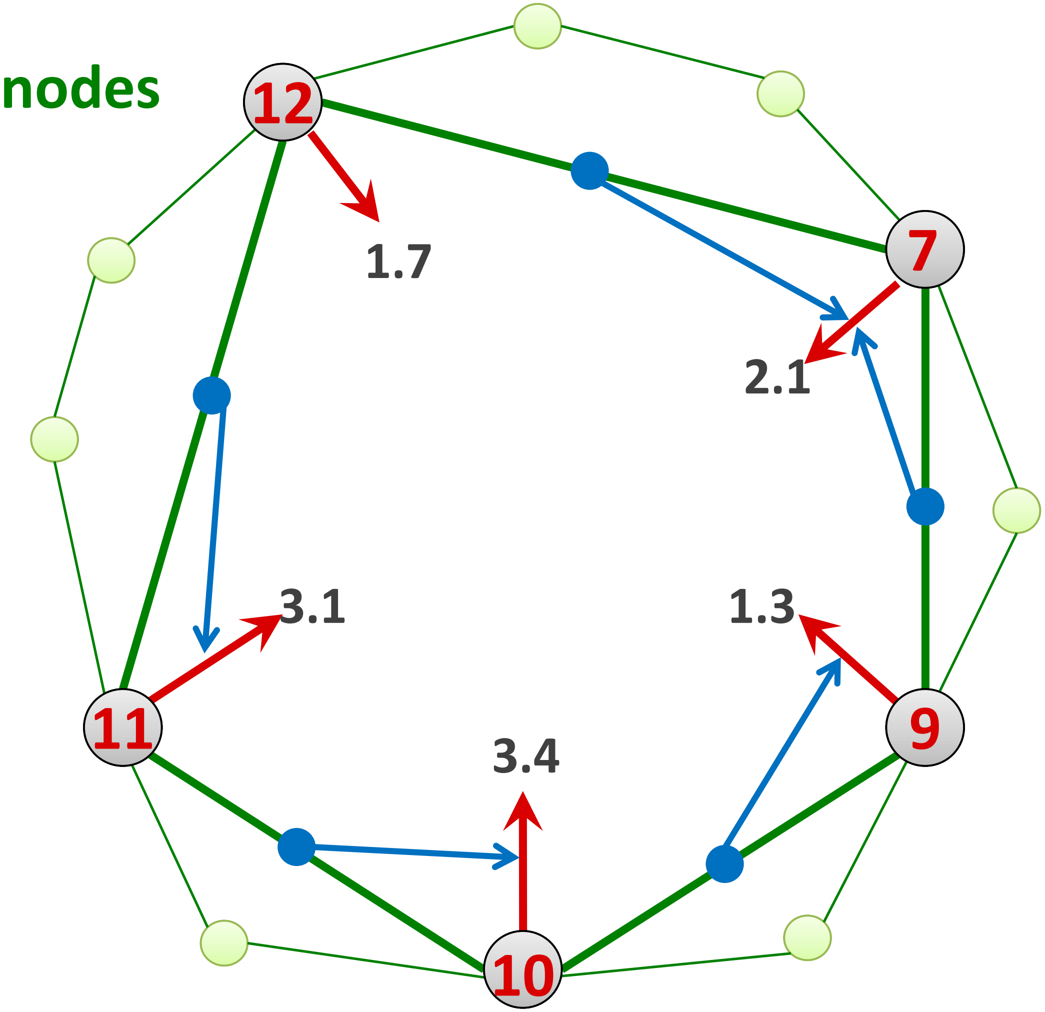
Metric TSP, Nearest Neighbor

Triangle Inequality:

optimal tour on remaining nodes

≤

overall optimal tour



$$\underbrace{\text{green}}_{\leq \text{OPT}} \geq \text{marked red}$$

marked red ≤ OPT

Metric TSP, Nearest Neighbor

Analysis works in phases:

- In each phase, assign each optimal edge to some greedy edge
  - Cost of greedy edge $\leq$ cost of optimal edge
- Each greedy edge gets assigned ≤ 2 optimal edges
  - At least half of the greedy edges get assigned
- At end of phase:

  Remove nodes for which greedy edge is assigned

  Consider optimal solution for remaining points
- **Triangle inequality:** remaining opt. solution $\leq$ overall opt. sol.

- Cost of greedy edges assigned in **each phase $\leq$ opt. cost**
- **Number of phases $\leq \log_2 n$**
  - +1 for last greedy edge in tour

Metric TSP, Nearest Neighbor

- Assume:

  NN: cost of greedy tour, OPT: cost of optimal tour

- We have shown:

$$\frac{\mathrm{NN}}{\mathrm{OPT}} \leq \underbrace{\overset{\text{last red edge}}{\overset{\downarrow}{1}} + \overset{\text{\#phases}}{\overset{\swarrow}{\log_2 n}}}_{\text{approximation ratio}}$$

$$(\mathrm{NN} \leq (1 + \log_2 n) \cdot \mathrm{OPT})$$

- Example of an **approximation algorithm**
- We will later see a ${}^{3}/{}_{2}$-approximation algorithm for metric TSP