Algorithm Theory

Chapter 2
Greedy Algorithms

Part IV:
The Greedy Algorithm for Matroids

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Matroids

- Same as MST, but more abstract…

Simple example:

$E := \{1, 2, 3, 4\}$

$I := \left\{ \begin{matrix} \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \\ \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\} \end{matrix} \right\}$

Matroid: **pair $(\boldsymbol{E}, \boldsymbol{I})$** (set system)

- $\boldsymbol{E}$: finite set, called the **ground set**
- $\boldsymbol{I}$: finite family of finite subsets of E (i.e., $I \subseteq 2^E$), called **independent sets**

(E, I) needs to satisfy 3 properties:

1. Empty set is independent, i.e., $\emptyset \in \boldsymbol{I}$ (implies that $I \neq \emptyset$)
2. **Hereditary property**: For all $A \subseteq E$ and all $A' \subseteq A$,

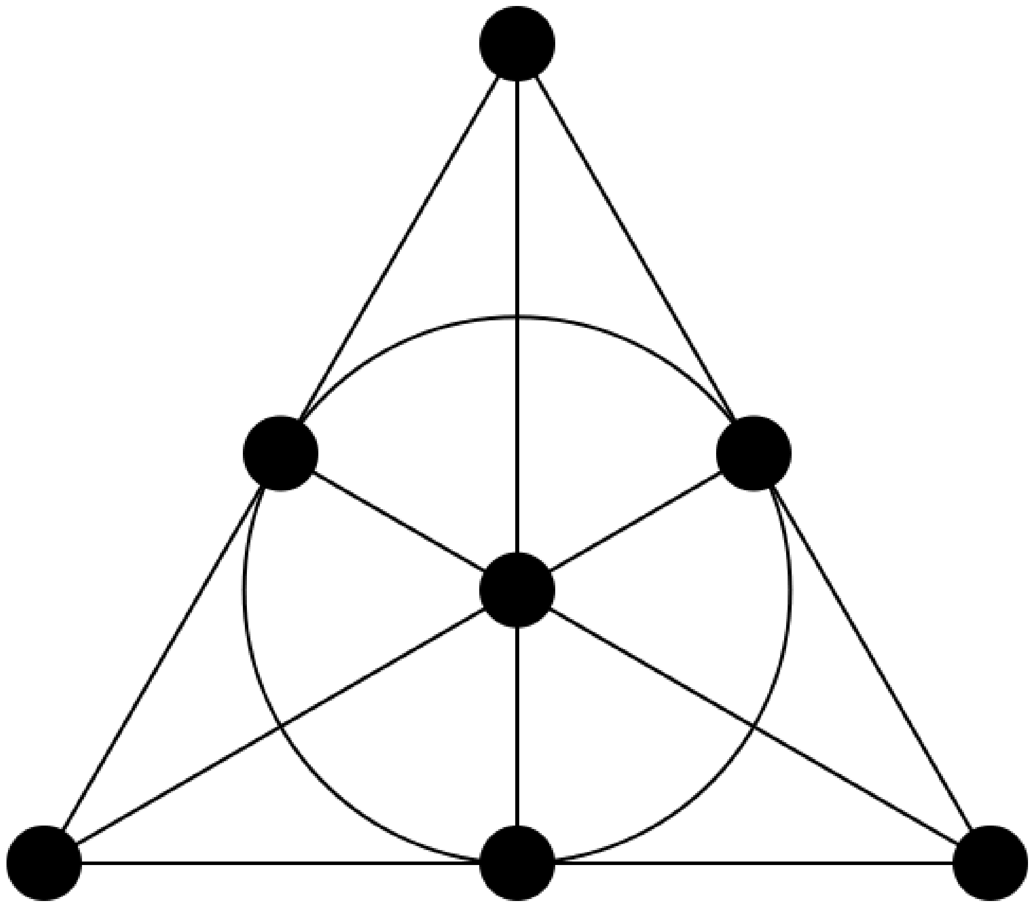
   if $\boldsymbol{A} \in \boldsymbol{I}$, then also $\boldsymbol{A}' \in \boldsymbol{I}$

3. **Augmentation / Independent set exchange property:**
   If $A, B \in I$ and $|A| > |B|$, there exists $x \in A \setminus B$ such that

   $$\mathbf{B}' := \boldsymbol{B} \cup \{\boldsymbol{x}\} \in \boldsymbol{I}$$

Example

- Fano matroid:
  - Smallest finite projective plane of order 2…



Matroids and Greedy Algorithms

Weighted matroid: each $e \in E$ has a weight $w(e) \geq 0$

- Recall that all independent sets in I consist of a finite set of elements of E.

Goal: find maximum weight independent set

Greedy algorithm:

1. Start with $S = \emptyset$
2. Add max. weight $x \in E \setminus S$ to S such that $S \cup \{x\} \in I$

Claim: greedy algorithm computes optimal solution

Greedy is Optimal

Matroid (E, I),
weights $w(x) \geq 0$ for all $x \in E$

- S: greedy solution — $S \subseteq E$, $S \in \mathcal{I}$
- A: any other solution (ind. set) — $A \subseteq E$, $A \in \mathcal{I}$

$|S| \geq |A|$: $(s \geq a)$

for contradiction, assume $|A| > |S|$: exch. prop: $\exists x \in A \setminus S$ s.t. $S \cup \{x\} \in \mathcal{I}$

greedy would have added x

$w(S) \geq w(A)$:

for contradiction, assume $w(S) < w(A)$

$S = \{x_1, x_2, \ldots, x_s\}$ $\quad w(x_1) \geq w(x_2) \geq \ldots \geq w(x_s)$

$A = \{y_1, y_2, \ldots, y_a\}$ $\quad w(y_1) \geq w(y_2) \geq \ldots \geq w(y_a)$

will show that (*)

$\forall i \in \{1, \ldots, a\}: w(x_i) \geq w(y_i)$

$\Rightarrow w(S) \geq w(A)$

$\neg(*) \Longrightarrow$ there is a smallest $k \leq a$ s.t. $w(x_k) < w(y_k)$

$S' = \{x_1, \ldots, x_{k-1}\}$ $\quad$ augm. prop.: $\exists y \in A' \setminus S'$ s.t. $S' \cup \{y\} \in \mathcal{I}$

$w(y) \geq w(y_k) > w(x_k)$

$A' = \{y_1, \ldots, y_k\}$ $\quad$ greedy considers y before x_k

$|A'| > |S'|$ $\quad$ greedy would add y

↯

Matroids: Examples

Forests of a graph $\boldsymbol{G = (V, E)}$:

- forest F: subgraph with no cycles (i.e., $F \subseteq E$)
- $\mathcal{F}$: set of all forests → $(E, \mathcal{F})$ is a matroid
- Greedy algorithm gives maximum weight forest
  - equivalent to MST problem

Bicircular matroid of a graph $\boldsymbol{G = (V, E)}$:

- $\mathcal{B}$: set of edges such that every connected subset has ≤ 1 cycle
- $(E, \mathcal{B})$ is a matroid → greedy gives max. weight such subgraph

Linearly independent vectors:

- Vector space V, E: finite set of vectors, I: sets of lin. indep. vect.
- Fano matroid can be defined like that

Forest Matroid of Graph $G = (V, E)$

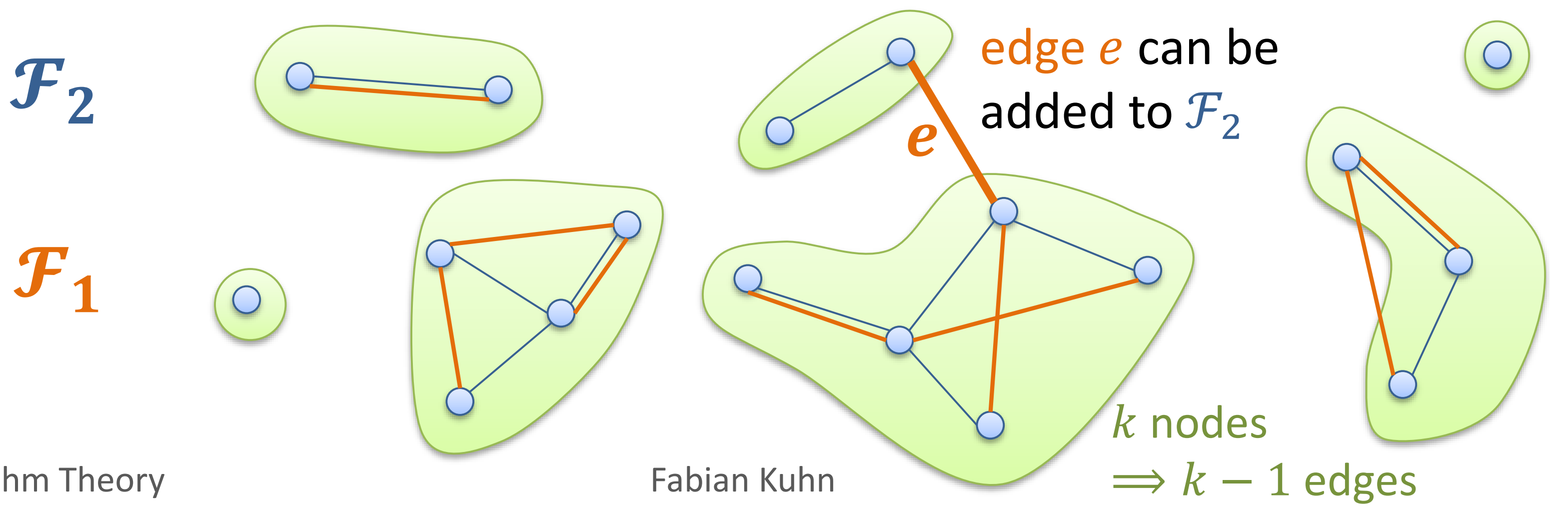
Ground set: E (edges) **Independent sets:** $\mathcal{F}$ (forests of G)

Basic properties: $\emptyset \in \mathcal{F}$ + hereditary property

- Empty graph has no cycles, removing edges doesn't create cycles

Independent set exchange property:

- Given $\mathcal{F}_1, \mathcal{F}_2$ s.t. $|\mathcal{F}_1| > |\mathcal{F}_2|$
  - $\exists e \in \mathcal{F}_1 \setminus \mathcal{F}_2$ s.t. $\mathcal{F}_2 \cup \{e\}$ is a forest
- $\mathcal{F}_1$ needs to have an edge e connecting two components of $\mathcal{F}_2$
  - Because it can only have $|\mathcal{F}_2|$ edges connecting nodes inside components



Greedoid

- Matroids can be generalized even more

- Relax hereditary property:

  Replace $A' \subseteq A \in I \implies A' \in I$

  by $\emptyset \neq A \in I \implies \exists a \in A,\ \text{s.t. } A \setminus \{a\} \in I$

- Augmentation property holds as before

- Under certain conditions on the weights, greedy is optimal for computing the max. weight $A \in I$ of a greedoid.
  - Additional conditions automatically satisfied by hereditary property

- More general than matroids