Algorithm Theory

Chapter 6
Graph Algorithms

Part VI:
Circulation

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Circulations with Demands

Given: Directed network with positive edge capacities

Sources & Sinks: Instead of one source and one destination, several sources that generate flow and several sinks that absorb flow.

Supply & Demand: sources have supply values, sinks demand values

Goal: Compute a flow such that source supplies and sink demands are exactly satisfied

- The circulation problem is a feasibility rather than a maximization problem

Circulations with Demands: Formally

Given: Directed network $G = (V, E)$ with

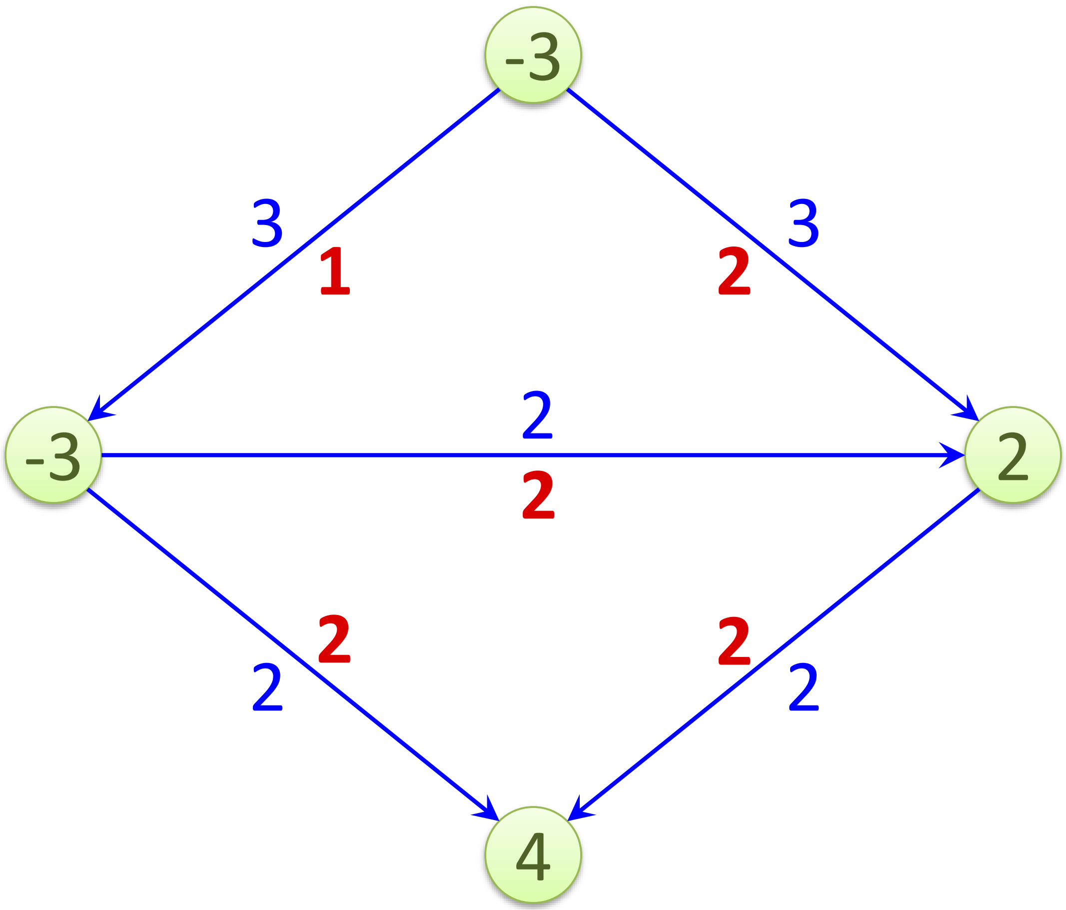
- Edge capacities $c_e \geq 0$ for all $e \in E$
- Node demands $d_v \in \mathbb{R}$ for all $v \in V$
  - $d_v > 0$: node needs flow and therefore is a sink
  - $d_v < 0$: node has a supply of $-d_v$ and is therefore a source
  - $d_v = 0$: node is neither a source nor a sink

Flow: Function $f: E \to \mathbb{R}_{\geq 0}$ satisfying

- *Capacity Conditions*: $\forall e \in E: \quad 0 \leq f(e) \leq c_e$
- *Demand Conditions*: $\forall v \in V: \quad f^{\text{in}}(v) - f^{\text{out}}(v) = d_v$

Objective: Does a flow f satisfying all conditions exist?
If yes, find such a flow f.

Example



Condition on Demands

Claim: If there exists a feasible circulation with demands d_v for $v \in V$, then

$$\sum_{v \in V} d_v = 0.$$

Proof:

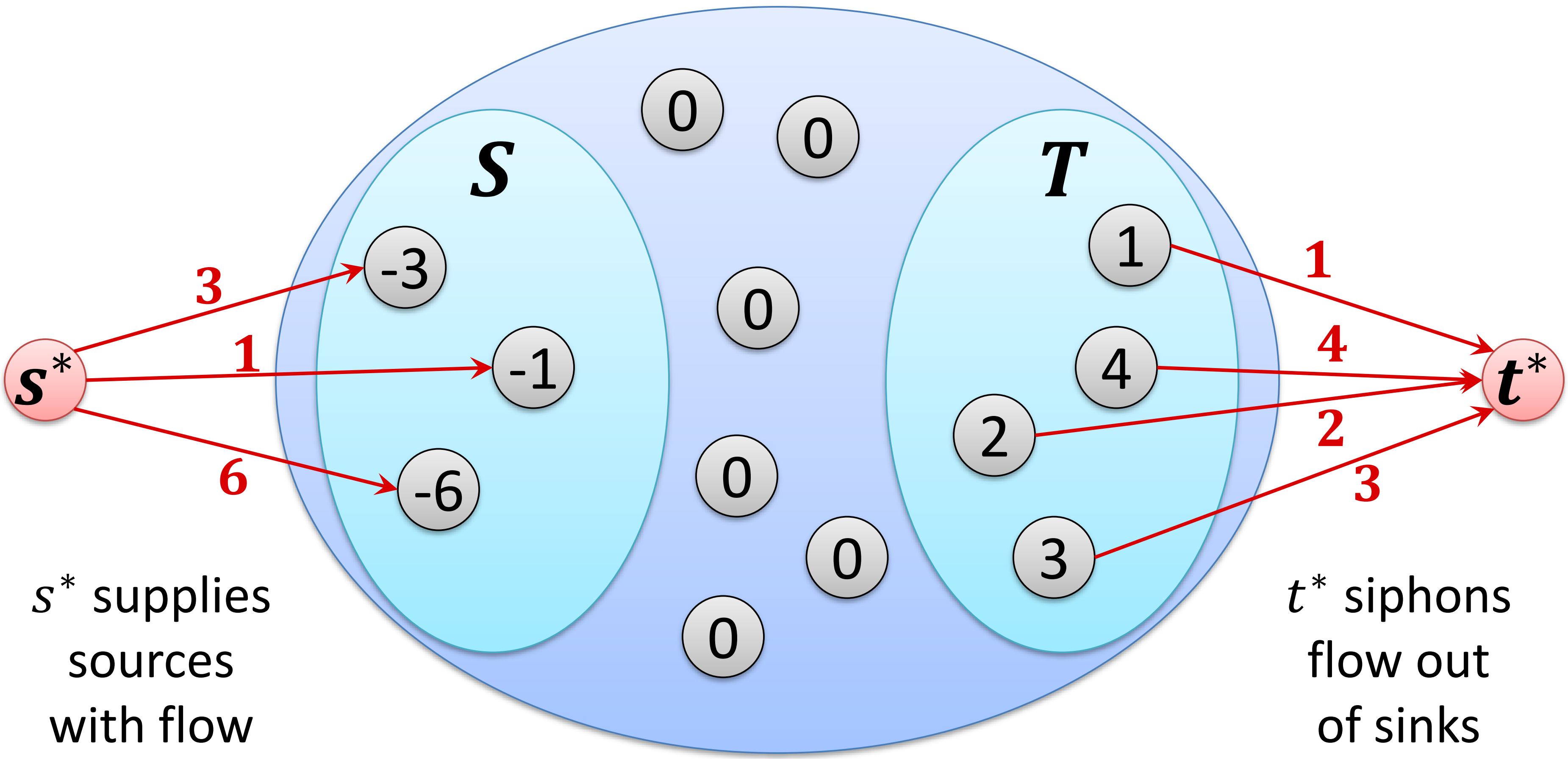
- $\sum_v d_v = \sum_v \left(f^{\text{in}}(v) - f^{\text{out}}(v)\right)$
- $f(e)$ of each edge e appears twice in the above sum with different signs → overall sum is 0

Total supply = total demand:

$$\text{Define } \boldsymbol{D} \coloneqq \sum_{v:d_v<0} -\boldsymbol{d_v} = \sum_{v:d_v>0} \boldsymbol{d_v}$$

Reduction to Maximum Flow

- Add "super-source" s^* and "super-sink" t^* to network

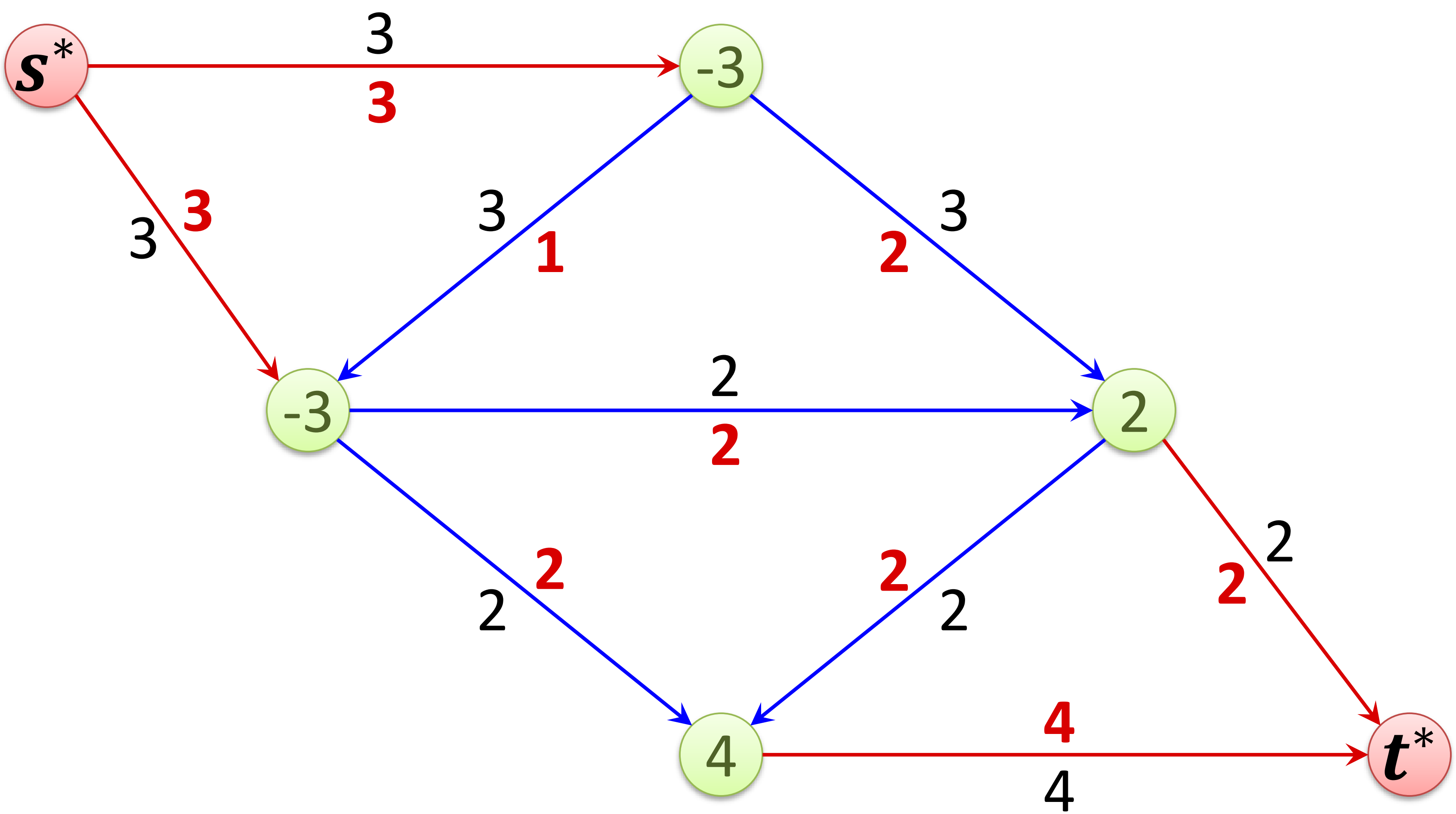


s^* supplies sources with flow

t^* siphons flow out of sinks

- valid circulations $\Leftrightarrow$ valid s^*-t^* flow that saturates all red edges.

Example



Formally…

Reduction: Get graph G' from graph as follows

- Node set of G' is $V \cup \{s^*, t^*\}$
- Edge set is E and edges
  - (s^*, v) for all v with $d_v < 0$, capacity of edge is $-d_v$
  - (v, t^*) for all v with $d_v > 0$, capacity of edge is d_v

Observations:

- Capacity of min s^*-t^* cut is at most D (e.g., the cut $(s^*, V \cup \{t^*\})$)
- A feasible circulation on G can be turned into a feasible flow of value D of G' by saturating all (s^*, v) and (v, t^*) edges.
- Any flow of G' of value D induces a feasible circulation on G
  - (s^*, v) and (v, t^*) edges are saturated
  - By removing these edges, we get exactly the demand constraints

Circulation with Demands

Theorem: There is a feasible circulation with demands d_v, $v \in V$ on graph G if and only if there is a flow of value D on G'.

- If all capacities and demands are integers, there is a valid integer circulation (if there is a valid circulation)

The max flow min cut theorem also implies the following:

Theorem: The graph G has a feasible circulation with demands d_v, $v \in V$ if and only if the sum of all demands is zero and for all cuts (A, B),

$$\sum_{v \in B} d_v \leq c(A, B).$$

Circulation: Demands and Lower Bounds

Given: Directed network $G = (V, E)$ with

- Edge capacities $c_e > 0$ and **lower bounds $\boldsymbol{0 \leq \ell_e \leq c_e}$ for $\boldsymbol{e \in E}$**
- Node demands $d_v \in \mathbb{R}$ for all $v \in V$
  - $d_v > 0$: node needs flow and therefore is a sink
  - $d_v < 0$: node has a supply of $-d_v$ and is therefore a source
  - $d_v = 0$: node is neither a source nor a sink

Flow: Function $f: E \to \mathbb{R}_{\geq 0}$ satisfying

- *Capacity Conditions*: $\forall e \in E$: $\boldsymbol{\ell_e \leq f(e) \leq c_e}$
- *Demand Conditions*: $\forall v \in V$: $f^{\text{in}}(v) - f^{\text{out}}(v) = d_v$

Objective: Does a flow f satisfying all conditions exist?
If yes, find such a flow f.

Solution Idea

- Define initial circulation $f_0(e) = \ell_e$
  Satisfies capacity constraints: $\forall e \in E: \ell_e \leq f_0(e) \leq c_e$

- Define

$$L_v := f_0^{\text{in}}(v) - f_0^{\text{out}}(v) = \sum_{e \text{ into } v} \ell_e - \sum_{e \text{ out of } v} \ell_e$$

- If $L_v = d_v$, demand condition is satisfied at v by f_0, otherwise, we need to superimpose another circulation f_1 such that

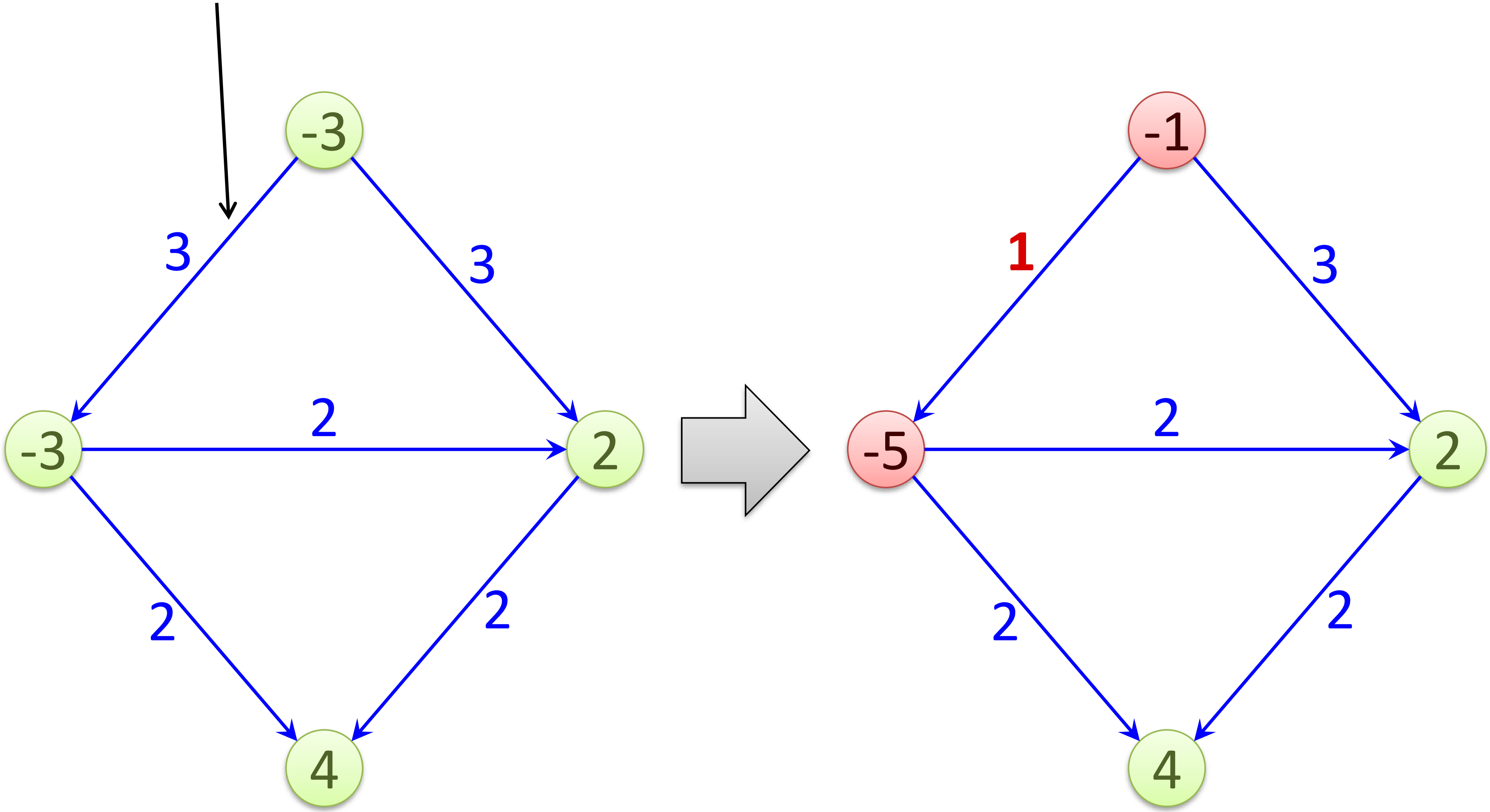
$$d_v' := f_1^{\text{in}}(v) - f_1^{\text{out}}(v) = d_v - L_v$$

- Remaining capacity of edge e: $c_e' := c_e - \ell_e$

- We get a circulation problem with new demands d_v', new capacities c_e', and no lower bounds

Eliminating a Lower Bound: Example

Lower bound of 2

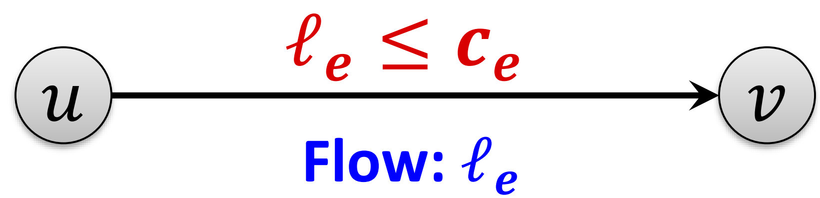


Reduce to Problem Without Lower Bounds

Graph $G = (V, E)$:

- Capacity: For each edge $e \in E$: $\ell_e \leq f(e) \leq c_e$
- Demand: For each node $v \in V$: $f^{\text{in}}(v) - f^{\text{out}}(v) = d_v$

Model lower bounds with supplies & demands:



Create Network G' (without lower bounds):

- For each edge $e \in E$: $c_e' = c_e - \ell_e$
- For each node $v \in V$: $d_v' = d_v - L_v$

Circulation: Demands and Lower Bounds

Theorem: There is a feasible circulation in G (with lower bounds) if and only if there is feasible circulation in G' (without lower bounds).

- Given circulation f' in G', $f(e) = f'(e) + \ell_e$ is circulation in G
  - The capacity constraints are satisfied because $f'(e) \leq c_e - \ell_e$
  - Demand conditions:

$$f^{\text{in}}(v) - f^{\text{out}}(v) = \sum_{e \text{ into } v} \big(\ell_e + f'(e)\big) - \sum_{e \text{ out of } v} \big(\ell_e + f'(e)\big)$$
$$= L_v + (d_v - L_v) = d_v$$

- Given circulation f in G, $f'(e) = f(e) - \ell_e$ is circulation in G'
  - The capacity constraints are satisfied because $\ell_e \leq f(e) \leq c_e$
  - Demand conditions:

$$f'^{\text{in}}(v) - f'^{\text{out}}(v) = \sum_{e \text{ into } v} (f\ (e) - \ell_e) - \sum_{e \text{ out of } v} (f\ (e) - \ell_e)$$
$$= d_v - L_v$$

Integrality

Theorem: Consider a circulation problem with integral capacities, flow lower bounds, and node demands. If the problem is feasible, then it also has an integral solution.

Proof:

- Graph G' has only integral capacities and demands
- Thus, the flow network used in the reduction to solve circulation with demands and no lower bounds has only integral capacities
- The theorem now follows because a max flow problem with integral capacities also has an optimal integral solution

- It also follows that with the max flow algorithms we studied, we get an integral feasible circulation solution.