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Theoretical Computer Science - Bridging Course
Exercise Sheet 12

Due: Sunday, 30th of January 2022, 23:59 pm

Exercise 1: Predicate Logic: Interpretations *(2+2+2 Points)*

In *predicate logic* or *first order* a formula φ is given with respect to a *signature* $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{R} \rangle$ which introduces the basic components that φ consists of. The components are: The variable symbols $\mathcal{V}$, constant symbols $\mathcal{C}$, function symbols $\mathcal{F}$ and the set of relation symbols $\mathcal{R}$. The elements of the sets $\mathcal{V}, \mathcal{C}, \mathcal{F}$ are used to formulate *terms*, while the relations in $\mathcal{R}$ compare terms with each other.[1]

These components must be combined in a *well-formed* manner with logical connectives ($\wedge, \vee, \neg$, etc.), quantifiers ($\forall, \exists$), and the '$\doteq$' symbol which is a relation that represents equality of terms. Consult the lecture for the detailed inductive definition of first order formulae.

As it was the case in propositional logic, we require *interpretations* in order to evaluate first order formulae to true or false. An interpretation $I = \langle \mathcal{D}, \cdot^I \rangle$ has a *domain* $\mathcal{D}$ that represents the set of all values that variables can assume.

Furthermore an interpretation has a mapping $\cdot^I$ which assigns constant symbols $\mathbf{c} \in \mathcal{C}$ a fixed value $\mathbf{c}^I \in \mathcal{D}$ from the domain. A function symbol $f \in \mathcal{F}$ is assigned an explicit function $f^I : \mathcal{D}^k \to \mathcal{D}$. A relation symbol $R \in \mathcal{R}$ is assigned an explicit relation $R^I \subseteq \mathcal{D}^k$. The parameter k is called *arity*.

If a formula φ has *free variables* (variables that are not bound by a quantifier: $\forall, \exists$) then I requires an additional variable *assignment function* $\alpha : \mathcal{V} \to \mathcal{D}$ assigning each free variable a value from the domain. An interpretation I is called a model of a first order formula φ, if an assignment function α *exists* (!) such that $\varphi^{I,\alpha}$ evaluates to true (see lecture for details on how to evaluate φ with I, α).

Evaluate the given formulae with the given interpretations. Make clear why or why not an interpretation is a model for the formula.

(a) $\varphi_1 := \forall x \exists y\, f(y) \doteq x$, $I_1 := \langle \mathbb{Z}, \cdot^{I_1} \rangle$, $I_2 := \langle \mathbb{Q}, \cdot^{I_2} \rangle$ where $f^{I_1}(a) := f^{I_2}(a) := 2 \cdot a$.

(b) $\varphi_2 := \forall x \exists y\, f(y, y) \doteq x$, $I_1 := \langle \mathbb{R}, \cdot^{I_1} \rangle$, $I_2 := \langle \mathbb{C}, \cdot^{I_2} \rangle$ where $f^{I_1}(a, b) := f^{I_2}(a, b) := a \cdot b$.

(c) $\varphi_3 := (\forall x\, f(x, z) \doteq x) \wedge (\forall x \exists y\, f(x, y) \doteq z)$, $I_1 := \langle \mathbb{R}, \cdot^{I_1} \rangle$, $I_2 := \langle \mathbb{R}, \cdot^{I_2} \rangle$ where $f^{I_1}(a, b) := a + b$, $f^{I_2}(a, b) := a \cdot b$. *Hint: First determine which z satisfies the first condition.*

Exercise 2: Predicate Logic: Is it a model? *(1+2+2 Points)*

Consider the following **first order** formulae

$$\begin{aligned}
\varphi_1 &:= \forall x R(x, x) \\
\varphi_2 &:= \forall x \forall y\, R(x, y) \to (\exists z R(x, z) \wedge R(z, y)) \\
\varphi_3 &:= \exists x \exists y\, (\neg R(x, y) \wedge \neg R(y, x))
\end{aligned}$$

over signature $\mathcal{S}$ where x, y, z are variable symbols and R is a binary predicate. Give an interpretation

[1] In the following the signature is given implicitly via the symbols that are given in a formula. By convention bold symbols $\mathbf{c}$ represent constants, lowercase letters f functions and capital letters R relations.

(a) I_1 which is a **model** of $\varphi_1 \wedge \varphi_2$.

(b) I_2 which is **no model** of $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$.

(c) I_3 which is a **model** of $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$.

Exercise 3: Predicate Logic: Construct Formulae *(1+1+1 Points)*

Let $\mathcal{S} = \langle \{x, y, z\}, \emptyset, \emptyset, \{R\} \rangle$ be a signature. Translate the following sentences of first order formula over $\mathcal{S}$ into idiomatic English. Use $R(x, y)$ as statement 'x is a part of y'.

(a) $\exists x \forall y R(x, y)$.

(b) $\exists y \forall x R(x, y)$.

(c) $\forall x \forall y \exists z (R(x, z) \wedge R(y, z))$

Exercise 4: Predicate Logic: Entailment *(2+2+2 Points)*

Let φ, ψ be first order formulae over signature $\mathcal{S}$. Similar to propositional logic, in predicate logic we write $\varphi \models \psi$ if every model of φ is also a model for ψ. We write $\varphi \equiv \psi$ if both $\varphi \models \psi$ and $\psi \models \varphi$. A *knowledge base* KB is a set of formulae. A model of KB is model for all formulae in KB. We write $KB \models \varphi$ if all models of KB are models of φ. Show or disprove the following entailments.

(a) $(\exists x\, R(x)) \wedge (\exists x\, P(x)) \wedge (\exists x\, T(x)) \models \exists x\, (R(x) \wedge P(x) \wedge T(x))$.

(b) $(\forall x \forall y\, f(x, y) \doteq f(y, x)) \wedge (\forall x\, f(x, \mathbf{c}) \doteq x) \models \forall x\, f(\mathbf{c}, x) \doteq x$.

(c) $(\forall x\, R(x, x)) \wedge (\forall x \forall y\, R(x, y) \wedge R(y, x) \rightarrow x \doteq y) \wedge (\forall x \forall y \forall z\, R(x, y) \wedge R(y, z) \rightarrow R(x, z))$
$\models \forall x \forall y\, R(x, y) \vee R(y, x)$.

Hint: Consider order relations. E.g., $a \leq b$ (a less-equal b) and $a|b$ (a divides b).