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Theoretical Computer Science - Bridging Course
Exercise Sheet 8

Due: Sunday, 9th of January 2022, 23:59 pm

Exercise 1: The Class $\mathcal{P}$ *(2+2 Points)*

$\mathcal{P}$ is the set of languages which can be decided by an algorithm whose runtime can be bounded by $p(n)$, where p is a polynomial and n the size of the respective input (problem instance). Show that the following languages ($\cong$ problems) are in the class $\mathcal{P}$. Since it is typically easy (i.e. feasible in polynomial time) to decide whether an input is well-formed, your algorithm only needs to consider well-formed inputs. Use the $\mathcal{O}$-notation to bound the run-time of your algorithm.

(a) 4-CLIQUE $:= \{\langle G\rangle \mid G$ has a *clique* of size at least $4\}$

(b) 5-VERTEXCOVER $:= \{\langle G\rangle \mid G$ has a *vertex cover* of size at most $5\}$.

Remarks:

- In both problems G is an undirected, simple graph.
- A *clique* in a graph $G = (V, E)$ is a set $C \subseteq V$ such that for all $u, v \in C : \{u, v\} \in E$.
- A *vertex cover* of $G = (V, E)$ is a subset $C \subseteq V$ of nodes, such that for all $\{u, v\} \in E$ it holds that $u \in C$ or $v \in C$.

Exercise 2: The Class $\mathcal{NP}$ *(3+3 Points)*

Show that the following problems (languages) are in class $\mathcal{NP}$.

(a) Given a graph $G = (V, E)$ and an integer k, it is required to determine whether G contains a clique of size at least k, hence consider the following problem:
CLIQUE$:=\{\langle G, k\rangle \mid G$ has a clique of size at least k $\}$.

(b) A *hitting set* $H \subseteq \mathcal{U}$ for a given universe $\mathcal{U}$ and a set $S = \{S_1, S_2, \ldots, S_m\}$ of subsets $S_i \subseteq \mathcal{U}$, fulfills the property $H \cap S_i \neq \emptyset$ for $1 \leq i \leq m$ (H 'hits' at least one element of every S_i).

Given a universe set $\mathcal{U}$, a set S of subsets of $\mathcal{U}$, and a positive integer k, it is required to determine whether $\mathcal{U}$ contains a hitting set of size at most k, hence consider the following problem:
HITTINGSET$:=\{\langle \mathcal{U}, S, k\rangle \mid$ universe $\mathcal{U}$ has subset of size $\leq k$ that *hits* all sets in $S \subseteq 2^{\mathcal{U}}\}$.[1]

Exercise 3: The Class $\mathcal{NPC}$ *(5+5 Points)*

Let L_1, L_2 be languages (problems) over alphabets Σ_1, Σ_2. Then $L_1 \leq_p L_2$ (L_1 is polynomially reducible to L_2), iff a function $f : \Sigma_1^* \to \Sigma_2^*$ exists, that can be calculated in polynomial time and

$$\forall s \in \Sigma_1^* : s \in L_1 \iff f(s) \in L_2.$$

[1] The power set $2^{\mathcal{U}}$ of some ground set $\mathcal{U}$ is the set of *all subsets* of $\mathcal{U}$. So $S \subseteq 2^{\mathcal{U}}$ is a collection of subsets of $\mathcal{U}$.

Language L is called $\mathcal{NP}$-hard, if *all* languages $L' \in \mathcal{NP}$ are polynomially reducible to L, i.e.

$$L \text{ is } \mathcal{NP}\text{-hard} \iff \forall L' \in \mathcal{NP} : L' \leq_p L.$$

The reduction relation '$\leq_p$' is transitive ($L_1 \leq_p L_2$ and $L_2 \leq_p L_3 \Rightarrow L_1 \leq_p L_3$). Therefore, in order to show that L is $\mathcal{NP}$-hard, it suffices to reduce a known $\mathcal{NP}$-hard problem $\tilde{L}$ to L, i.e. $\tilde{L} \leq_p L$.
Finally a language is called $\mathcal{NP}$-complete ($\Leftrightarrow$: $L \in \mathcal{NPC}$), if

1. $L \in \mathcal{NP}$ and
2. L is $\mathcal{NP}$-hard.

(a) Show $\textsc{Clique} := \{\langle G, k\rangle \,|\, G$ has a clique of size at least $k\ \} \in \mathcal{NPC}$.

(b) Show $\textsc{HittingSet} := \{\langle \mathcal{U}, S, k\rangle \mid$ universe $\mathcal{U}$ has subset of size at most k that *hits* all sets in $S \subseteq 2^{\mathcal{U}}\} \in \mathcal{NPC}$.

For both parts, use the fact that $\textsc{VertexCover} := \{\langle G, k\rangle \mid$ Graph G has a *vertex cover* of size at most $k\} \in \mathcal{NPC}$.

Hint: For the poly. transformation ($\leq_p$) you have to describe an algorithm (with poly. run-time!) that transforms:
for part (a), an instance $\langle G, k\rangle$ of $\textsc{VertexCover}$ *into an instance $\langle G', k'\rangle$ of* $\textsc{Clique}$ *s.t. a vertex cover of size $\leq k$ in G becomes a clique of G' of size $\geq k'$ vice versa(!)*
for part (b), an instance $\langle G, k\rangle$ of $\textsc{VertexCover}$ *into an instance $\langle \mathcal{U}, S, k'\rangle$ of* $\textsc{HittingSet}$*, s.t. a vertex cover of size $\leq k$ in G becomes a hitting set of $\mathcal{U}$ of size $\leq k'$ for S and vice versa(!).*