University of Freiburg
Department of Computer Science
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Exam Theoretical Computer Science - Bridging Course

Monday, March 4, 2019, 14:00-15:30

Name: ....

Matriculation No.: ....

Signature: ....

Do not open or turn until told so by the supervisor!

- Write your **name** and **matriculation number** on this page and sign the document!
- Write your name on **all sheets**!
- Your **signature** confirms that you feel physically and mentally able to write the exam and that you have answered all questions without any help.
- Write legibly and only use a pen (ink or ball point). Do **not use red**! Do **not use a pencil**!
- This is an **open book exam** therefore printed or hand-written material is allowed.
- However, **no electronic devices** are allowed.
- There are **eight tasks** (with several sub-tasks each) and there is a **total of 90 points**.
- **35 points are sufficient** in order to pass the exam. **70 points** are sufficient to get the best mark.
- Only **one solution per task** is considered! Make sure to strike out alternative solutions, otherwise the one yielding the minimal number of points is considered.
- **Detailed steps** might help you to get more points in case your final result is incorrect.
- The keywords **Show...** or **Prove...** indicate that you need to prove or explain your answer carefully.
- The keywords **Give...** or **State...** indicate that you only need to provide a plain answer.
- You may use information given in a **Hint** without explaining them.
- **Read each task thoroughly** and make sure you understand what is expected from you.
- **Raise your hand** if you have a question regarding the formulation of a task.
- **Use the space below each task and the back of the sheet for your solution.** The last two sheets of this exam are blank and can be used for solutions. If you need additional sheets, raise your hand.

Question	1	2	3	4	5	6	7	**Total**
Points								
Maximum	10	19	10	14	10	12	15	90

Task 1: Basic Mathematical Skills (10 Points)

1. A *tree* is a simple, connected graph without cycles. Show that a tree with n nodes has $n-1$ edges. *(5 Points)*

   Hint: You may use that a tree has at least one leaf, i.e., a node of degree one.

2. Let $T_1 = (V, E_1)$, $T_2 = (V, E_2)$, ..., $T_k = (V, E_k)$ be k trees on the same set of vertices V of size n (assume that n is even). Let $G = (V, E_1 \cup E_2 \cup \ldots \cup E_k)$ be the union of these trees. Show that more than half of the nodes of G have degree at most $4k$ in G. *(5 Points)*

Sample Solution

1. Induction over the number of nodes:

   Induction base: A tree with 1 node has 0 edges.

   Induction step: Assume the statement holds for n. Let T be a tree with $n+1$ nodes. Let v be a leaf of T and $T' := T \setminus \{v\}$. Then T' is a tree with n nodes and has by assumption $n-1$ edges. As v is a leaf (i.e., has only one incident edge), it follows that T has n edges.

2. By the previous exercise we know that G has at most $k(n-1)$ edges. However, if half the nodes or more have degree at least $4k$ there would be at least $\frac{1}{2}\frac{n}{2}4k = nk$ edges, a contradiction.

Task 2: Regular Languages (19 Points)

1. Let $\Sigma = \{a, b, \ldots, z\}$ be the set of letters from the English alphabet. Let L be the language over Σ consisting of all words that appear in the book "Harry Potter and the Chamber of Secrets". Is L regular? Explain your answer in one sentence. *(2 Points)*

2. Let $\Sigma = \{a, b\}$. Let L_1 be the language defined by the regular expression $a^*b^*a^*$ and L_2 the language defined by a^*b^*b. *(7 Points)*

   Draw a DFA for L_1, L_2, and $L_1 \setminus L_2 := \{w \in \Sigma^* \mid w \in L_1 \text{ and } w \notin L_2\}$.

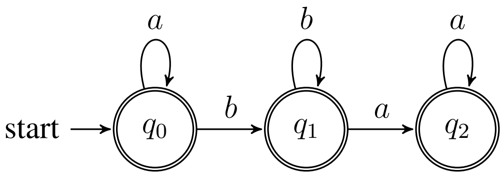
3. Show that if L and L' are regular languages over some alphabet Σ, then also $L \setminus L'$ is regular. *(3 Points)*

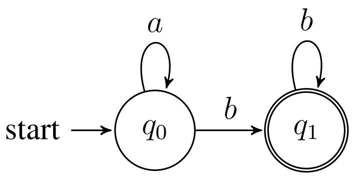
4. Use the pumping lemma to show that $L = \{w \in \{a, b\}^* \mid w \text{ contains more } a\text{'s than } b\text{'s}\}$ is not regular. *(7 Points)*

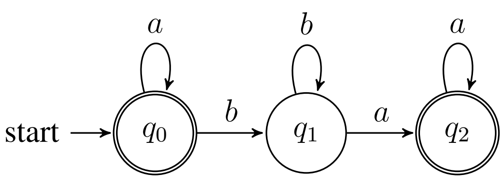
Sample Solution

1. Yes, finite languages are regular.

2.







3. Let L and L' be regular languages. We have

$$L \setminus L' = L \cap \overline{L'} = \overline{\overline{L} \cup L'}\,.$$

   As regular languages are closed under union and complement, $L \setminus L'$ is also regular.

4. Assume L was regular and p the pumping length. Consider the word $s = a^p b^{p-1} \in L$. Then there are x, y, z such that $s = xyz$, $|y| > 0$, $|xy| \leq p$ and $xy^0 z = xz \in L$. From $|xy| \leq p$ it follows that y only consists of a's. As $|y| > 0$, xz has at most as many a's as b's and is therefore not contained in L, a contradiction.

Task 3: Context-Free Languages (10 Points)

Give a context-free grammar that generates the language

$$\{a^i b^j c^k \mid i = j \text{ or } j = k \text{ where } i, j, k \geq 0\} .$$

Hint: It is maybe helpful to remember that context-free languages are closed under union.

Sample Solution

$S \rightarrow S_1 \mid S_2$

$S_1 \rightarrow DC$

$D \rightarrow aDb \mid \varepsilon$

$C \rightarrow Cc \mid \varepsilon$

$S_2 \rightarrow AE$

$E \rightarrow bEc \mid \varepsilon$

$A \rightarrow Aa \mid \varepsilon$

Task 4: Decidability (14 Points)

1. Show that $A = \{\langle R, S\rangle \mid R$ and S are regular expressions and $L(R) \subseteq L(S)\}$ is decidable. *(8 Points)*

2. Show that $EQ_{TM} = \{\langle M_1, M_2\rangle \mid M_1, M_2$ are Turing Machines and $L(M_1) = L(M_2)\}$ is undecidable. *(6 Points)*

   Hint: You may use that $E_{TM} = \{\langle M\rangle \mid M$ is a Turing Machine and $L(M) = \emptyset\}$ is undecidable.

Sample Solution

1. Let T be the Turing Machine deciding the language $\{\langle D\rangle \mid D$ is a DFA with $L(D) = \emptyset\}$ (known from the lecture). We have $L(R) \subseteq L(S) \Leftrightarrow L(R)\setminus L(S) = \emptyset$. Thus we construct a decider for A in the following way:

   On input $\langle R, S\rangle$ where R, S are regular expression:

   - Convert R and S into equivalent DFAs (like in the lecture)
   - Construct a DFA D for the regular language $L(R) \setminus L(S) = \overline{\overline{L(R)} \cup L(S)}$
   - Run T on input $\langle D\rangle$. Accept iff T accepts.

2. Assume we had a TM R that decides EQ_{TM}. We construct a decider for E_{TM}:

   On input $\langle M\rangle$ where M is a TM:

   - Construct a TM B that rejects all inputs.
   - Run R on $\langle M, B\rangle$. Accept iff R accepts.

Task 5: $\mathcal{O}$ - Notation **(10 Points)**

State whether the following claims are true or false *(1 point each)*. Then **prove or disprove** the claim. Use the definition of the $\mathcal{O}$-notation.

1. $(\ln n)^2 \in \mathcal{O}\left(\ln(n^2)\right)$ *(1+4 Points)*
2. $3n^2 + 8n \in \mathcal{O}\left(n^2\right)$ *(1+4 Points)*

Sample Solution

1. The claim is false. For any $c > 0$, there is a n_0 such that $\ln n > 2c$ for all $n \geq n_0$ which implies that $(\ln n)^2 = \ln n \cdot \ln n > 2c \ln n = c \ln n^2$ for all $n \geq n_0$.
2. The claim is true. Choose $c = 11$. Then for all $n \geq 1$ we have $n \leq n^2$ and thus $3n^2 + 8n \leq 3n^2 + 8n^2 = 11n^2$

Task 6: Complexity (12 Points)

Given a set U of n elements ('universe') and a collection $S \subseteq \mathcal{P}(U)$ of subsets of U, a selection $C_1, \ldots, C_k \in S$ of k sets is called a *set cover* of (U, S) of size k if $C_1 \cup \ldots \cup C_k = U$.

Show that the problem

$$\text{SETCOVER} := \{\langle U, S, k\rangle \,|\, U \text{ is a set, } S \subseteq \mathcal{P}(U) \text{ and there is a set cover of } (U, S) \text{ of size } k\}$$

is NP-complete.

You may use that

$$\text{DOMINATINGSET} = \{\langle G, k\rangle \mid G \text{ has a dominating set with } k \text{ nodes}\}.$$

is NP-complete. A subset of the nodes of a graph G is a *dominating set* if every other node of G is adjacent to some node in the subset.

Sample Solution

SETCOVER is in NP: Guess a collection $C_1, \ldots, C_k \in S$ of k sets from S. Go through all elements of U and check if it is in one of the C_i. This takes polynomial time.

SETCOVER is NP-hard: We reduce DOMINATINGSET to SETCOVER. Let $G = (V, E)$ be a graph and k an integer. We define a SETCOVER instance in the following way: We choose V to be the universe, i.e., $U = V$ and $S := \{\Gamma_G(v) \mid v \in V\}$. This conversion takes polynomial time. Then $\Gamma_G(v_1), \ldots, \Gamma_G(v_k)$ is a set cover of (U, S) iff $v_1, \cdots, v_k$ is a dominating set of G. Hence, $\langle U, S, k\rangle \in$ SETCOVER iff $\langle G, k\rangle \in$ DOMINATINGSET.

Task 7: Logic (15 Points)

1. Consider the following propositional formula

$$\psi := (x \lor y \to \bot) \land (z \to x \land w) \land (y \lor z).$$

   Either find a satisfying assignment for ψ or use the resolution calculus to show that ψ is unsatisfiable. *(9 Points)*

2. Consider the following first order logical formulae

$$\begin{aligned} \varphi_1 &:= \ \forall x \ \neg R(x,x) \\ \varphi_2 &:= \ \forall x \, \forall y \ (x \neq y \to R(x,y) \lor R(y,x)) \\ \varphi_3 &:= \ \exists x \, \forall y \ (x \neq y \to R(x,y)) \end{aligned}$$

   where x, y are variable symbols and R is a binary predicate. Give an interpretation

   (a) I_1 which is a model of $\varphi_1 \land \varphi_2 \land \varphi_3$. *(3 Points)*

   (b) I_2 which is a model of $\varphi_1 \land \varphi_2 \land \neg\varphi_3$. *(3 Points)*

   Remark: No proof required.

Sample Solution

1. ψ is unsatisfiable. ψ is equivalent to $\neg x \land \neg y \land (\neg z \lor x) \land (\neg z \lor w) \land (y \lor z)$ which is equivalent to the knowledge base $\{\{\neg x\}, \{\neg y\}, \{\neg z, x\}, \{\neg z, w\}, \{y, z\}\}$.

$$\begin{aligned} \{\neg x\}, \{\neg z, x\} &\vdash_{\mathbf{R}} \{\neg z\} \\ \{\neg y\}, \{y, z\} &\vdash_{\mathbf{R}} \{z\} \\ \{\neg z\}, \{z\} &\vdash_{\mathbf{R}} \Box \end{aligned}$$

2. (a) Take $I_1 := (\mathbb{N}, R^{I_1})$ where $R^{I_1}(x,y) :\Leftrightarrow x <_{\mathbb{N}} y$.

   (b) Take $I_2 := (\mathbb{Z}, R^{I_1})$ where $R^{I_1}(x,y) :\Leftrightarrow x <_{\mathbb{Z}} y$.