Theoretical Computer Science (Bridging Course)

Mathematical Preliminaries

Gian Diego Tipaldi

UNI FREIBURG

Mathematical Set

- Collection of distinct objects
- Description of a set

$$\mathcal{A} = \{1, 3, 4\}$$

$$\mathcal{B} = \left\{x^2 \mid x \in \mathbb{N}, n \leq 20\right\}$$

- Empty set $\varnothing = \{\}$
- Set membership

$$3 \in \mathcal{A} \qquad\qquad 5 \notin \mathcal{B}$$

Special Sets

- Subset and proper subset

$$\{3, 4, 1\} \subseteq \mathcal{A} \qquad \{1, 2, 4\} \subset \mathcal{B}$$

- Properties: $\varnothing \subset \mathcal{S}$ $\qquad \mathcal{S} \subseteq \mathcal{S}$
- Power set: the set of all subsets

$$\mathcal{A} = \{1, 3, 4\}$$

$$P(\mathcal{A}) = \{\varnothing, \{1\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}\}$$

- Cartesian product between sets

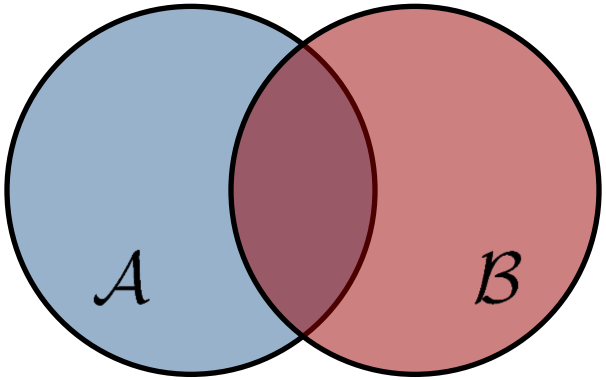
$$\mathcal{A} = \{1, 3, 4\} \qquad \mathcal{B} = \{a, b\}$$

$$\mathcal{A} \times \mathcal{B} = \{\{1, a\}, \{1, b\}, \{3, a\}, \{3, b\}, \{4, a\}, \{4, b\}\}$$

Set Operations – Union

- Union is "similar" to addition

$$\{6,7\} \cup \{8,9\} = \{6,7,8,9\}$$
$$\{3,4,1\} \cup \{3,5\} = \{3,4,1,5\}$$



$\mathcal{A} \cup \mathcal{B}$

- Some properties

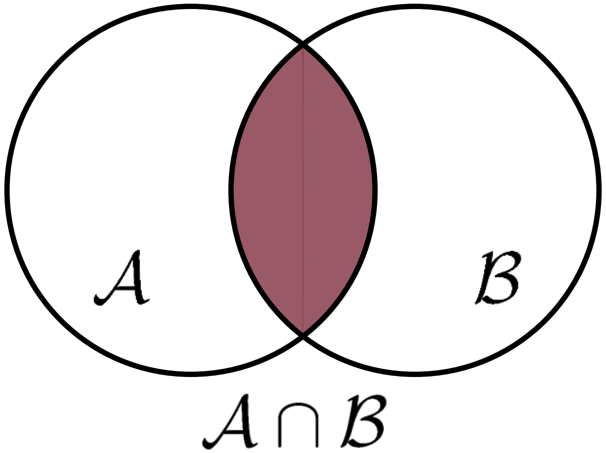
$$\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}$$
$$\mathcal{A} \cup (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C}$$
$$\mathcal{A} \subseteq \mathcal{A} \cup \mathcal{B}$$
$$\mathcal{A} \cup \emptyset = \mathcal{A}$$
$$\mathcal{A} \cup \mathcal{A} = \mathcal{A}$$
$$\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow \mathcal{A} \cup \mathcal{B} = \mathcal{B}$$

Set Operations – Intersection

- Intersction "takes" the common part

$$\{6,7\} \cap \{8,9\} = \varnothing$$

$$\{3,4,1\} \cap \{3,5\} = \{3\}$$



- Some properties

$$\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A}$$

$$\mathcal{A} \cap (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C}$$

$$\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}$$

$$\mathcal{A} \cap \varnothing = \varnothing$$

$$\mathcal{A} \cap \mathcal{A} = \mathcal{A}$$

$$\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow \mathcal{A} \cap \mathcal{B} = \mathcal{A}$$

Mathematical Sequence

- Collection of objects with an order
- Description of a sequence

$$\mathcal{A} = (1, 3, 4, 3)$$

$$\mathcal{B} = (a_n)_{k=1}^{20} \quad a_k = k^2$$

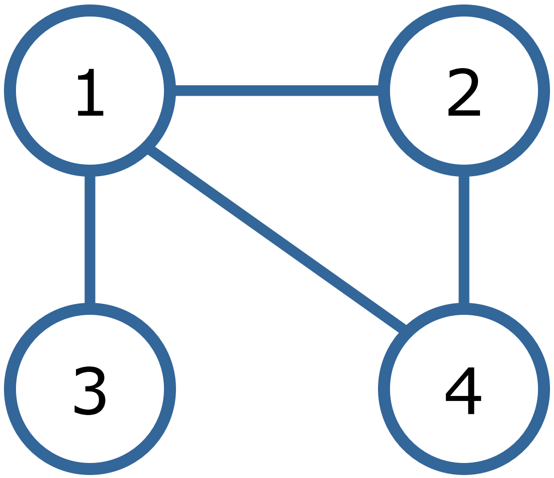
- Finite sequences
  - K elements -> k-tuples
  - 2 elements -> pair
- Infinite sequences

Graph

- Represents objects and relations
- Ordered pair of Vertices and Edges

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

$$\mathcal{G} = (\{1,2,3,4\}, \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,4\}\})$$



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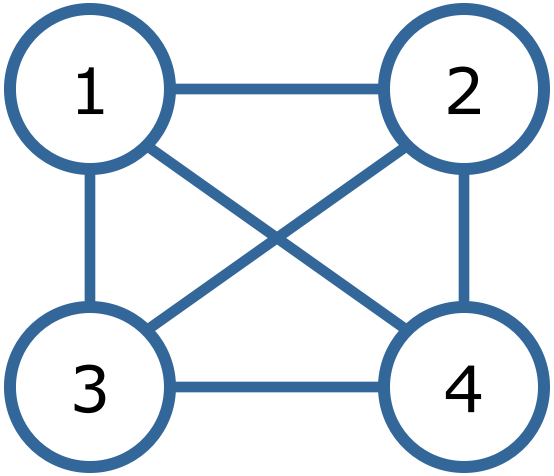
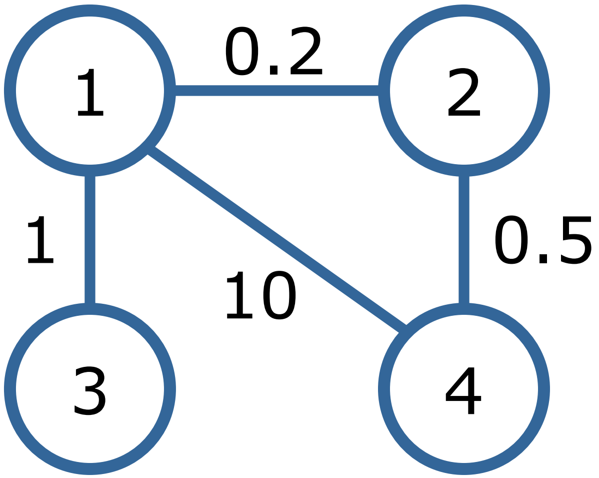


Weighted Graph

- Relations are “measurable”
- Associate a number with the edges

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

$$\mathcal{G} = (\{1,2,3,4\}, \{(0.2, \{1,2\}), (1, \{1,3\}), (10, \{1,4\}), (0.5, \{2,4\})\})$$



Subgraph

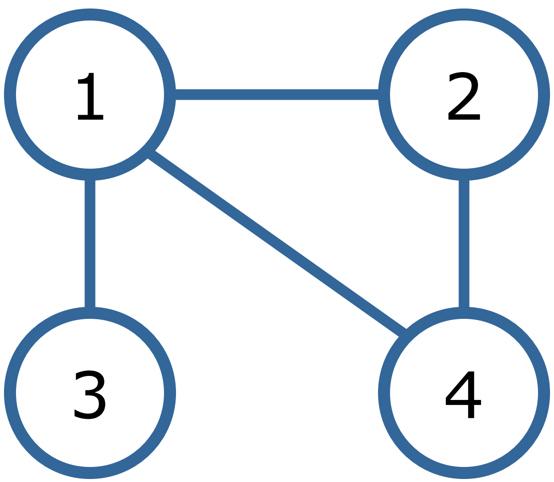
- Subset of vertices and edges

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

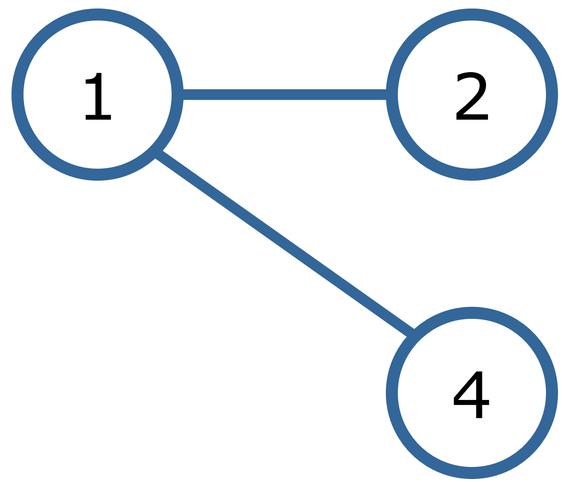
$$\mathcal{G} = (\{1,2,3,4\}, \{\{1,2\},\{1,3\},\{1,4\},\{2,4\}\})$$

$$\mathcal{S} = (\{1,2,4\}, \{\{1,2\},\{1,4\}\})$$

$\mathcal{G}$



$\mathcal{S}$



Induced Subgraph

- Subset of vertices and all their edges

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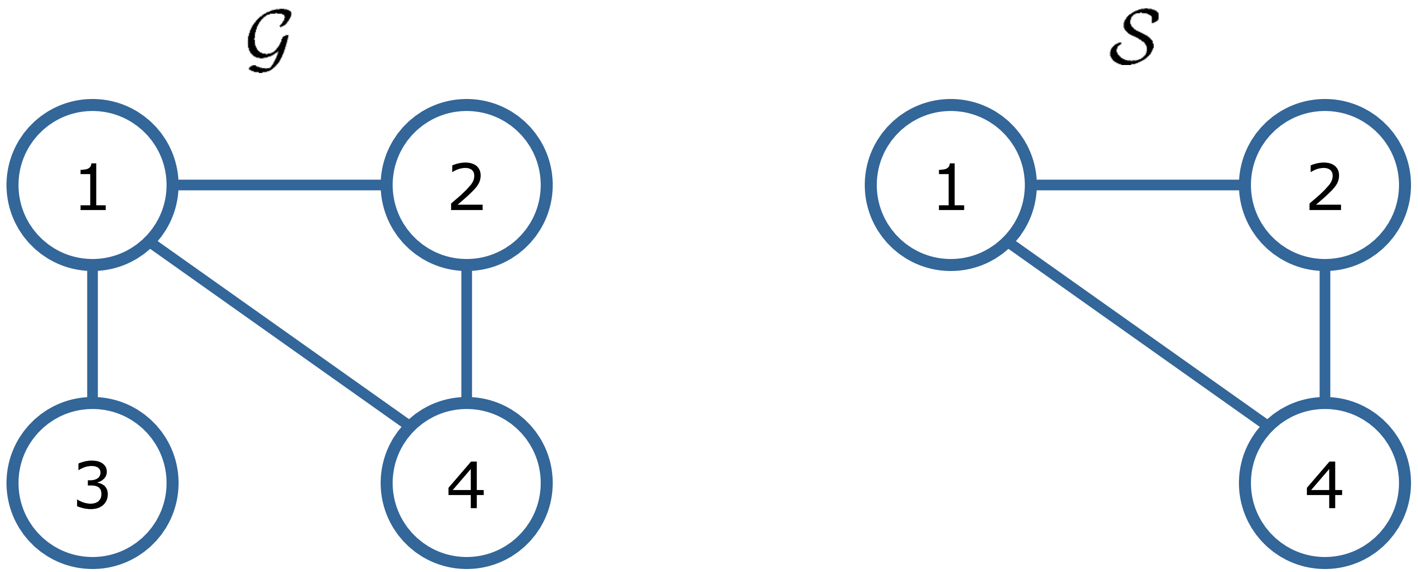
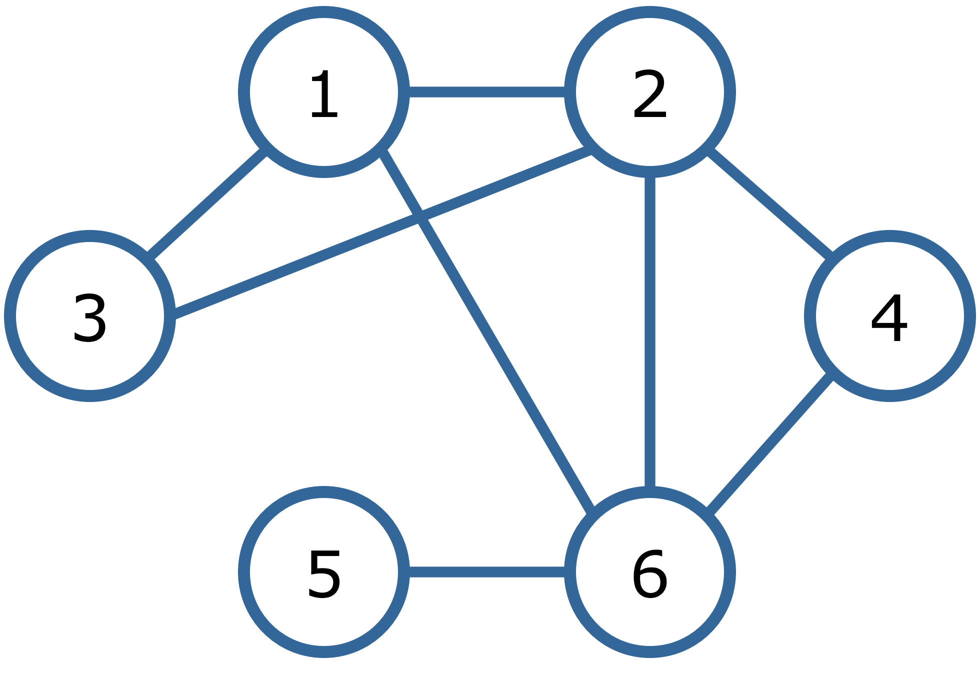


Paths in a Graph

- A sequence of vertices and their edges
- No vertices nor edges are repeated



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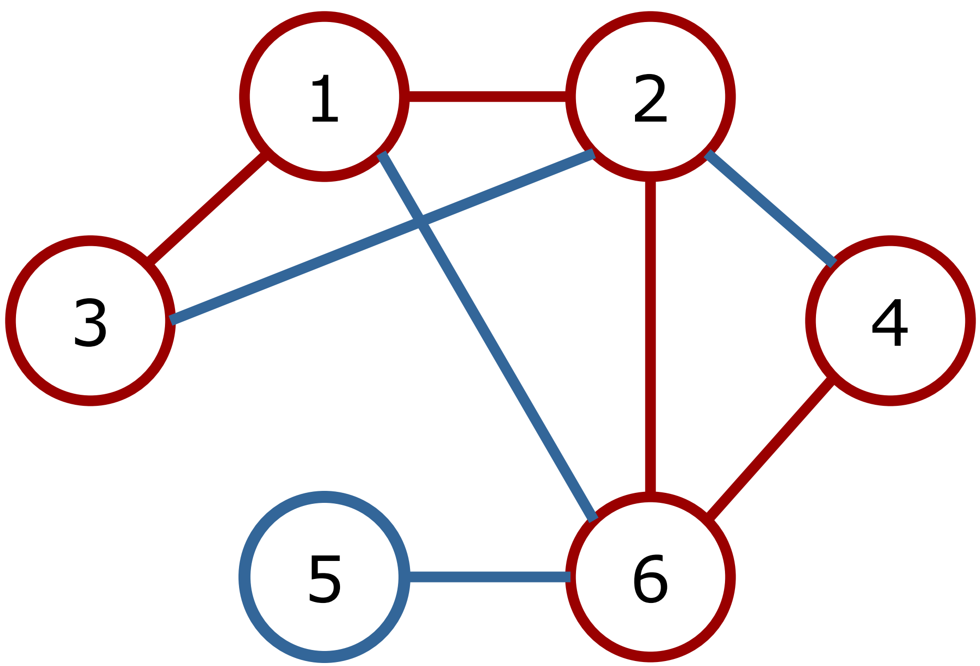

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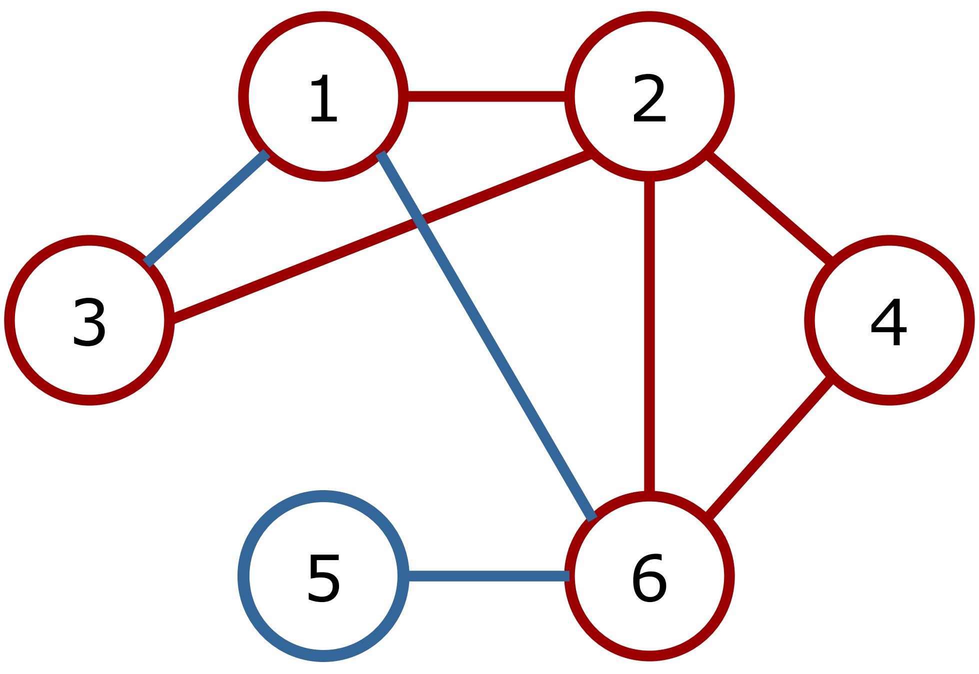

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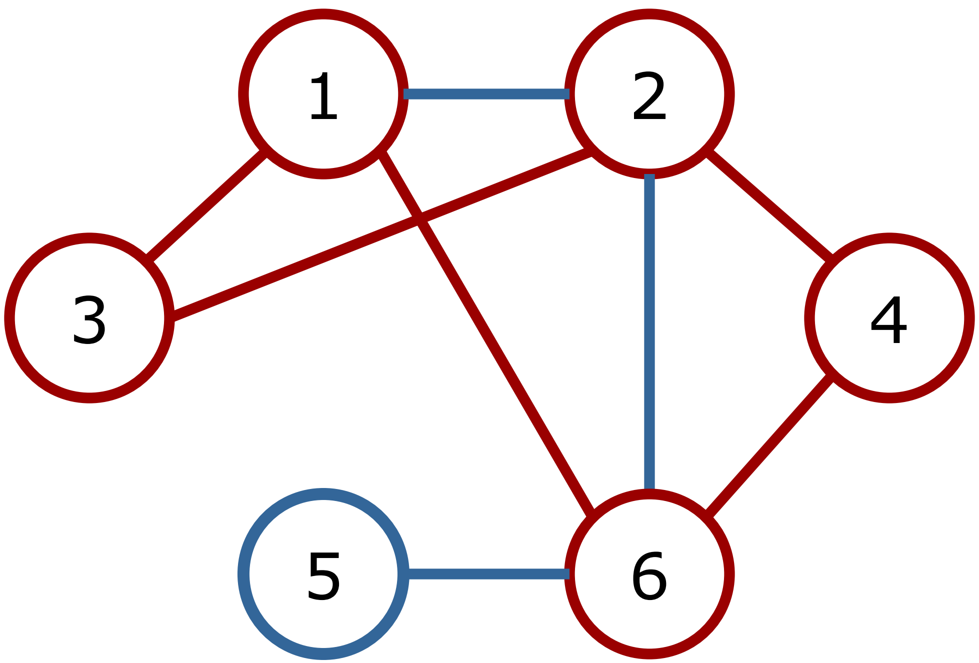
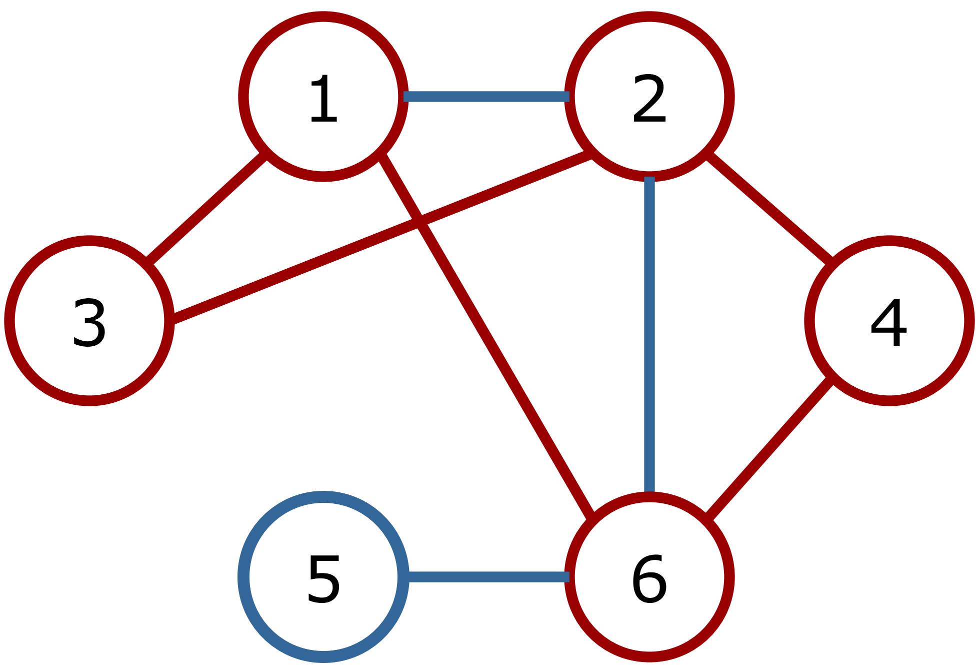


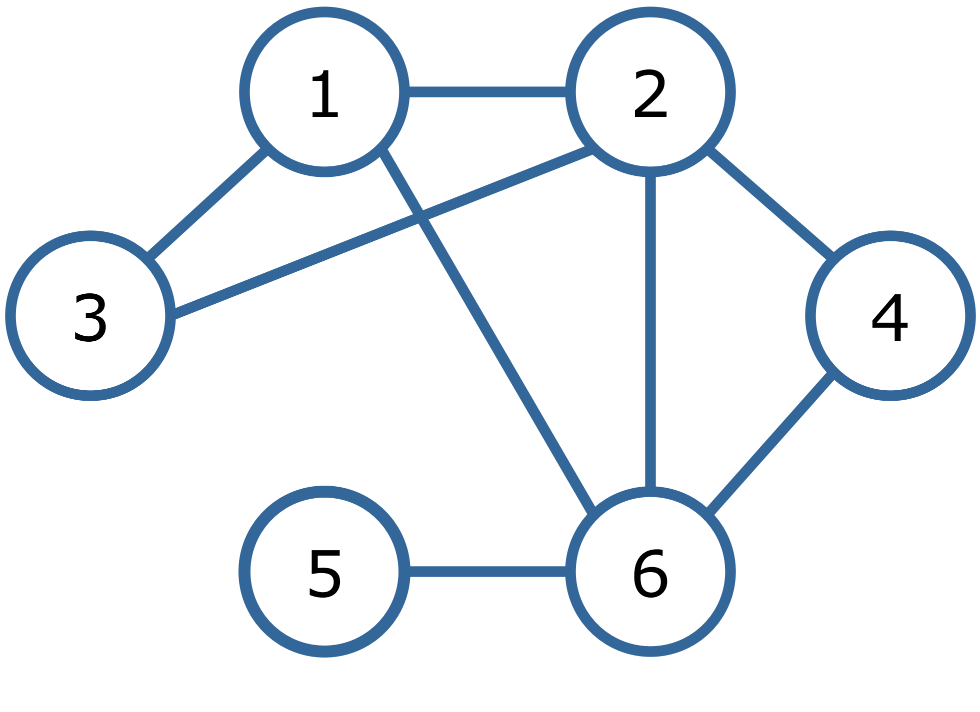
Cycles in a Graph

- A closed path in a graph
- First and last vertex is the same



Trees

- A tree is a special graph
- No cycles are present in a tree



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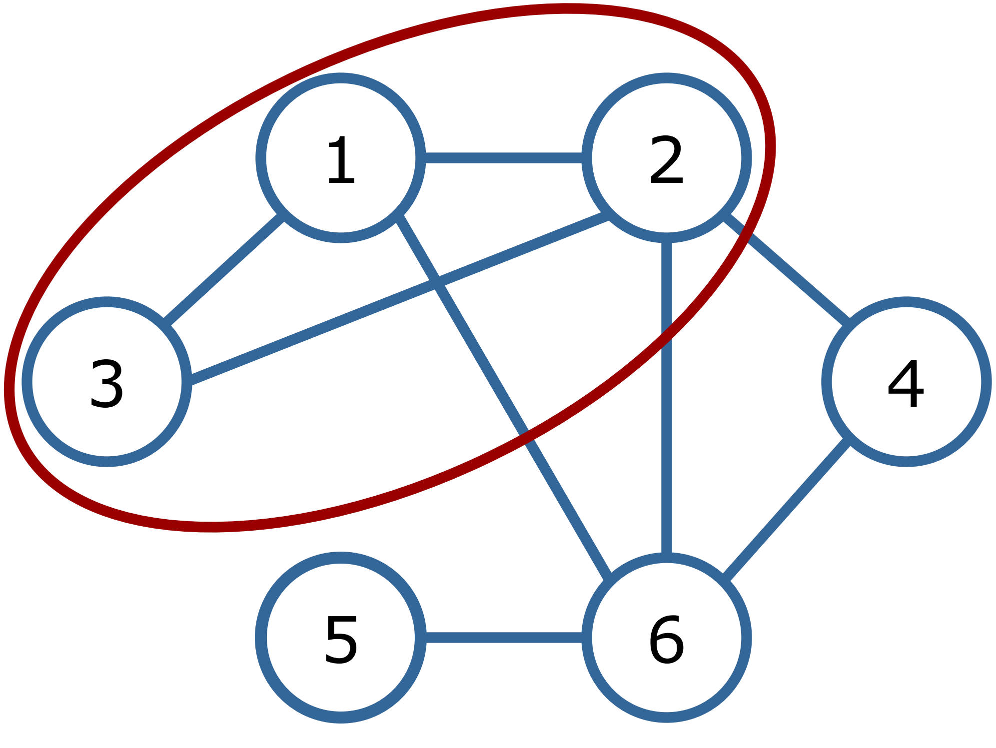

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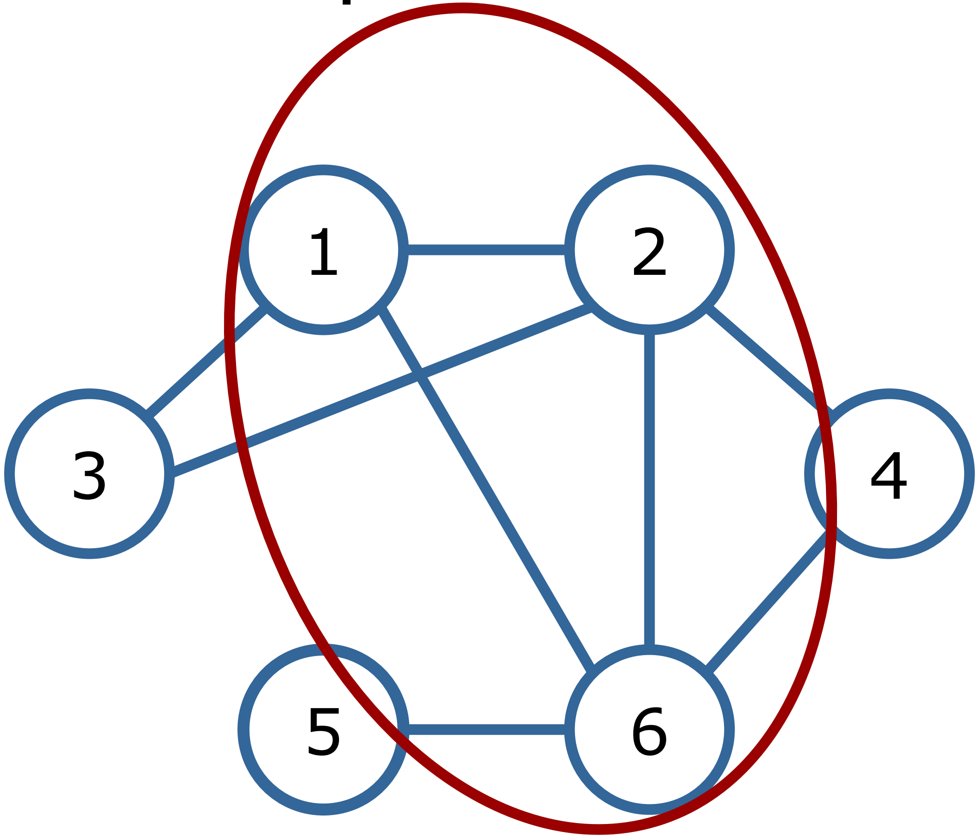

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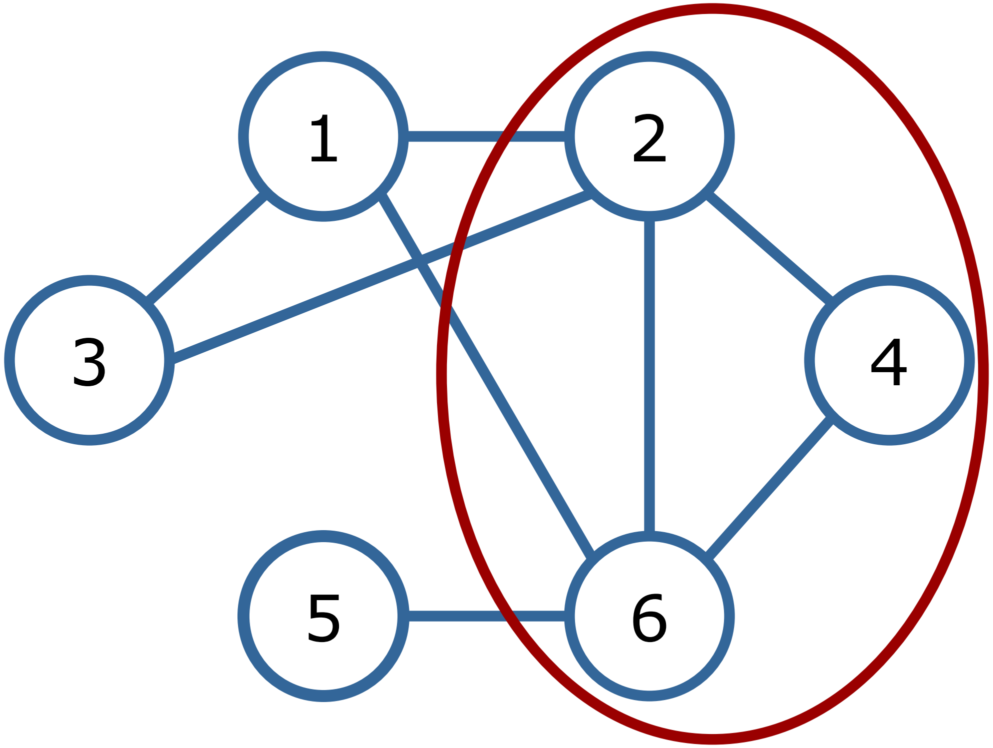

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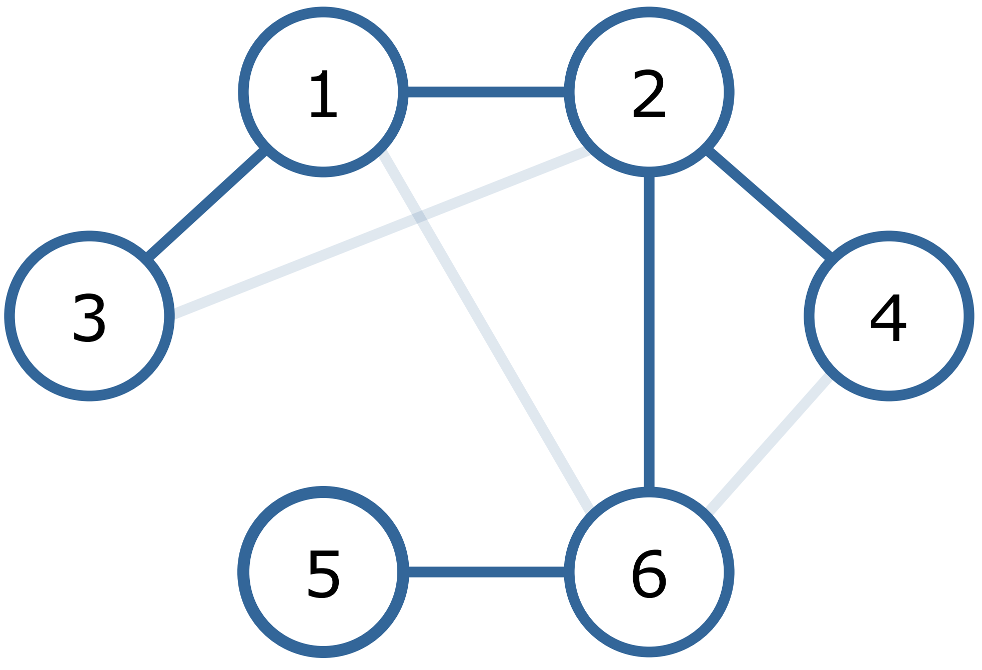
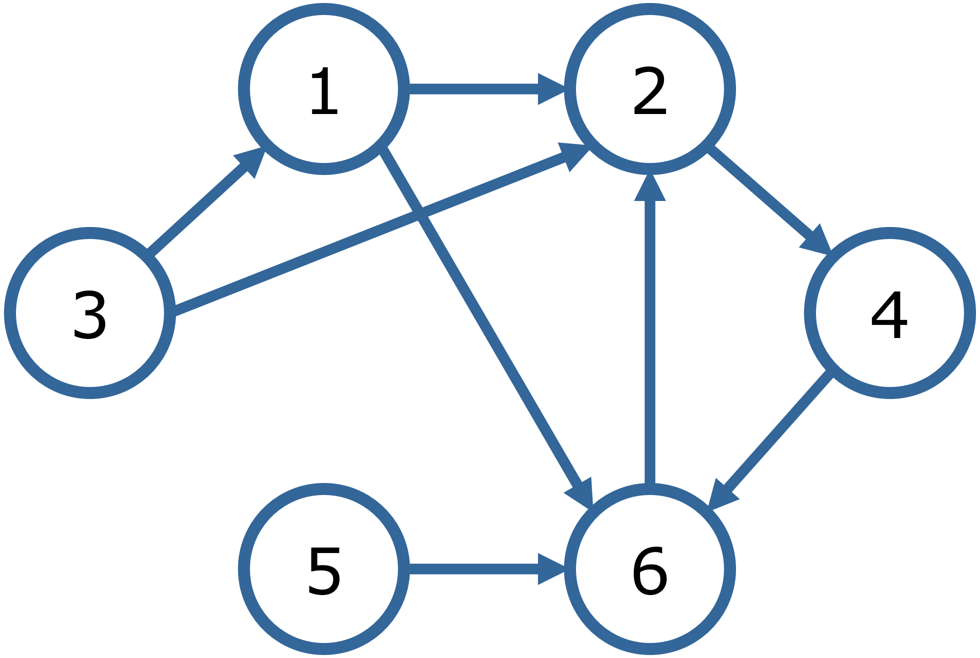


Directed Graph

- Edges are not sets, but ordered pairs

$$\mathcal{G} = (\{1,2,3,4,5,6\},\ \{\ (1,2),(3,1),(1,6),(2,4),(6,2),(3,2),(4,6),(5,6)\})$$



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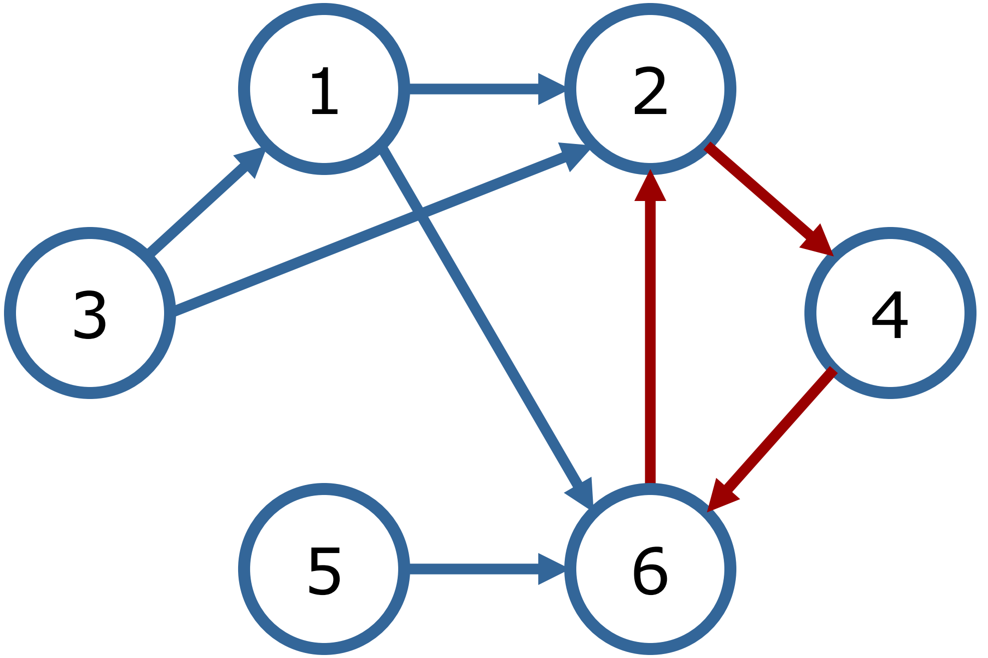
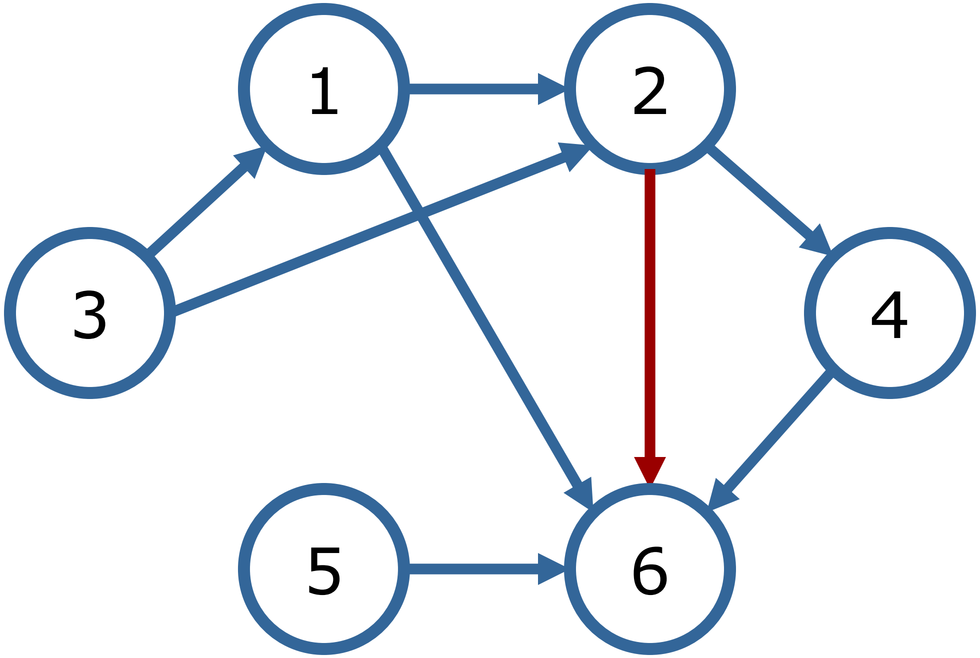


Directed Acyclic Graph

- Directed graph with no cycles

$$\mathcal{G} = (\{1,2,3,4,5,6\},\ \{\ (1,2),(3,1),(1,6),(2,4),\ (2,6),(3,2),(4,6),(5,6)\})$$



Strings and Languages

- An alphabet is a set of symbols

$$\Sigma = \{a, b, c, d\}$$

- A string is a sequence of symbols

$$s = aabbccddddddcccaaa$$

- Length of string = number of symbols
- aabb is a substring of ccaabbbbddd
- xy is the concatenation of x and y
- A language is a set of strings

Mathematical Proofs

- Direct proof
- Proof by construction/counterexample
- Proof by contradiction
- Proof by induction

- Formal enough to be convincing to your audience

Direct Proof

- Derive conclusions from premises

- Start from your assumptions
- Use logic to derive conclusions
- Tricky, must go through definitions

- Hint: try to think “backwards”

Direct Proof

- Let a,b,c be integers
- If a|b and b|c, then a|c

- a|b implies it exists k1, s.t. a = k1*b
- b|c implies it exists k2, s.t. b = k2*c
- We get then that a = k1*k2*c
- It exists k = k1*k2, s.t. a = k*c
- This impies a|c

Proof by Construction

- Prove that a particular object exists
- Demonstrate how to construct it
- Alternatively, find a counterexample

- All shapes that have four sides of equal length are squares
- Counterexample: Rhombi

Proof by Construction

- For all even numbers n>2, there exists a 3-regular graph with n vertices

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

$$\mathcal{V} = \{1, \cdots, n\}$$

$$\mathcal{E} = \{\{i, i+1\} \mid 1 \leq i \leq n-1\} \cup \{\{1, n\}\} \cup \{\{i, i+n/2\} \mid 1 \leq i \leq n/2\}$$

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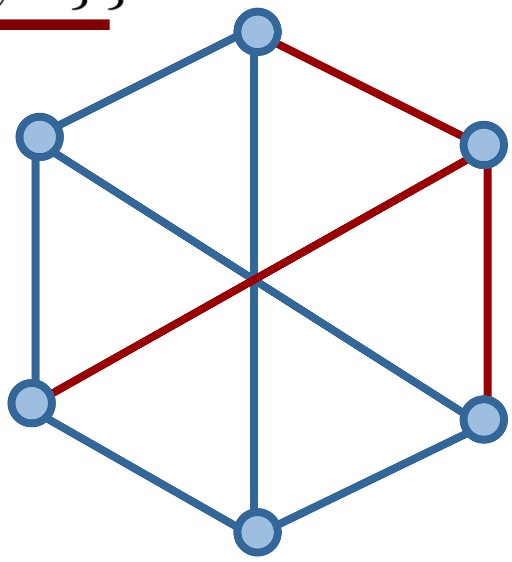

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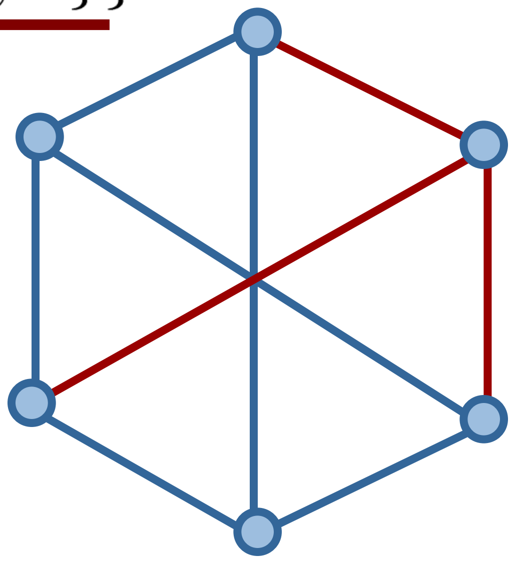

- **Why do we need n>2?**

Proof by Contradiction

- Assume the theorem is not true
- Show that it leads to a contradiction
- It violates the premises
- It violates some postulates/axioms
- Hence, the theorem must be true

- Example: $\sqrt{2}$ is irrational

Proof by Contradiction

- Assume $\sqrt{2}$ is rational

$$\sqrt{2} = \frac{a}{b} \quad \text{where } a \text{ and } b \text{ are integers and } \frac{a}{b} \text{ is reduced}$$

$$2 = \frac{a^2}{b^2}$$

$$2b^2 = a^2 \quad a^2 \text{ is even, hence } a = 2c \text{ is even}$$

$$2b^2 = 4c^2$$

$$b^2 = 2c^2 \quad b^2 \text{ is even, hence } b = 2d \text{ is even}$$

$$\frac{a}{b} = \frac{2c}{2d} \quad \frac{a}{b} \text{ is not reduced, contradiction}$$

Proof by Induction

- Prove a statement for a set of objects

- Base: prove it for a “small” object
- Induction: prove it for “bigger” objects assuming it holds for “smaller” ones

- Natural numbers
- Inductively defined objects

Inductively Defined Objects

- Objects are created by “adding” parts
- Object definition is recursive

Example: Rooted binary trees

- Base: A node is a tree
- Induction:
  - T1 and T2 are rooted binary trees
  - Take a node N, it is the new root
  - Add edges from N to T1 and T2

Proof by Induction

- Theorem:
  - A binary tree with n leaves has 2n-1 nodes
- Base:
  - A tree with one leaf has 2*2-1 = 1 node
  - A one leaf tree is a single node tree

Proof by Induction

- Induction:
  - Take a tree T with two subtrees U and V
  - Assume the theorem holds for U and V
  - U has x leaves and 2x - 1 nodes
  - V has y leaves and 2y - 1 nodes
  - T has z = x + y leaves
  - T has 2x - 1 + 2y - 1 + 1 =
    2(x+y) - 1 =
    2z – 1 nodes

Summary

- Sets, subsets, power sets
- Graphs and subgraphs
- Strings and languages
- Mathematical proofs
  - Direct
  - Construction
  - Contradiction
  - Induction