University of Freiburg
Dept. of Computer Science
Prof. Dr. F. Kuhn
G. Schmid

Algorithms and Datastructures
Winter Term 2023
Sample Solution Exercise Sheet 11

Due: Wednesday, Jan 31st, 12 pm

Exercise 1: Bitstrings without consecutive ones *(10 Points)*

Given a positive integer n, we want to compute the number of n-digit bitstrings without consecutive ones (e.g., for $n = 3$ this number is 5, as 000, 001, 010, 100, 101 are the 3-digit bitstrings without consecutive ones).

(a) Give an algorithm which solves this problem in time $\mathcal{O}(n)$. Explain the runtime. *(5 Points)*

(b) Implement your solution. You may use the template `DP.py`. Run your algorithm on the values 10, 20 und 50 and write your results in `erfahrungen.txt`. *(5 Points)*

Sample Solution

(a) Let $A(n, i)$ be the number of n-digit bitstrings without consecutive ones ending on i, for $i \in \{0, 1\}$. We have $A(n, 0) = A(n-1, 0) + A(n-1, 1)$ and $A(n, 1) = A(n-1, 0)$. It follows $A(n, 0) = A(n-1, 0) + A(n-2, 0)$ and for the base cases $A(1, 0) = 1$ and $A(2, 0) = 3$. The recursive structure is the same as for the Fibonacci numbers. The calculation therefore goes along similar lines as in the lecture (week 11, slide 6). The number of n-digit bitstrings without consecutive ones is $A(n + 1, 0)$.

(b) Cf. `DP.py` (another inplementation than described in (a)). The values for 10, 20 and 50 are 144, 17711 and 32951280099.

Exercise 2: Partitioning *(10 Points)*

Given a set $X = \{x_0, \ldots, x_{n-1}\}$ with $x_i \in \mathbb{N}$, we want to determine whether *there is* a subset $S \subseteq X$ such that $\sum_{x \in S} x = \sum_{x \in X \setminus S} x$ (it is not necessary to compute S).

(a) Let $W := \sum_{x \in X} x$. Give a recursive formula $s : \{0, \ldots, n-1\} \times \{0, \ldots, W\} \to \{$`True, False`$\}$ such that $s(i, j) =$ `True` if and only if there is a $S \subseteq \{x_0, \ldots, x_i\}$ such that $\sum_{x \in S} x = j$. Explain how s can be used to solve the above problem in time $\mathcal{O}(W \cdot n)$. *(5 Points)*

(b) Implement your solution. You may use the template `DP.py`. Run your algorithm on the sets given in `set1.txt`, `set2.txt` and `set3.txt` and write your results to `erfahrungen.txt` *(5 Points)*

Sample Solution

(a) Assume there is a set $S \subseteq \{x_0, \ldots, x_i\}$ with $\sum_{x \in S} x = j$. Then we either have $x_i \in S$ or $x_i \notin S$. In the first case, there is a set $S' \subseteq \{x_0, \ldots, x_{i-1}\}$ with $\sum_{x \in S'} x = j - x_i$. In the second case there is a set $S'' \subseteq \{x_0, \dots, x_{i-1}\}$ with $\sum_{x \in S''} x = j$. If such a set S does not exist, there is neither S' nor S''. We therefore have that S exists if and only if S' or S'' exist. We recursively define

$$s(i,j) = s(i-1, j-x_i) \vee s(i-1, j)$$

where $\vee$ is the or-operator which is true if one of the arguments is true. We define the following base cases. We set $s(i,0) =$ `True` since the empty set sums up to 0. We set $s(0,j) =$ `True` if and only if $x_0 = j$. We set $s(i,j) =$ `False` if $i < 0$ or $j < 0$.

If there is a set $S \subseteq X$ with $\sum_{x \in S} x = \sum_{x \in X \setminus S} x$, then both sums must equal $W/2$. We therefore obtain a solution of the problem by computing $s(n-1, W/2)$.

We apply dynamic programming to compute $s(n-1, W/2)$. As i and j only decrease in the recursion, we only have $n \cdot (W/2+1) = \mathcal{O}(n \cdot W)$ different possibilities for parameters (i,j). We therefore have to compute $\mathcal{O}(n \cdot W)$ the value of $s(i,j)$.

Computing a single value via $s(i,j) = s(i-1, j-x_i) \vee s(i-1,j)$ *without* the costs for the recursion takes $\mathcal{O}(1)$. We save all values $s(i,j)$ in a dictionary `memo[i,j]` and therefore have to compute the value $s(i,j)$ only once. As runtime we obtain $\mathcal{O}(n \cdot W)$.

(b) Cf. `DP.py`. The results for the sets `set1.txt`, `set2.txt` and `set3.txt` are `True, False, True`.