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Algorithm Theory
Exercise Sheet 1

Due: Friday, 27th of October, 2023, 10:00 am

Exercise 1: Landau and Recursions *(8 Points)*

(a) Is $\left(\log(\sqrt{n})\right)^2 \in \Theta(\log n)$ true? Justify. *(2 Points)*

Consider two real square matrices X and Y each of size $n \times n$. The goal is to find the matrix multiplication of X and Y i.e. $Z = X \times Y$, which is also of size $n \times n$.

(b) Use the principle of divide and conquer to design an algorithm that finds Z in time $O(n^3)$ and analyze its running time.
Hint: try dividing the matrices into submatrices of size $\frac{n}{2} \times \frac{n}{2}$. *(3 Points)*

(c) Let $X = \begin{pmatrix} A & L \\ G & O \end{pmatrix}$ and $Y = \begin{pmatrix} T & H \\ E & O' \end{pmatrix}$, where A, L, G, O, T, H, E, O' are matrices of size $\frac{n}{2} \times \frac{n}{2}$.
Let $Z = \begin{pmatrix} ? + P_4 - P_2 + ? & ? + ? \\ ? + ? & ? + P_5 - P_3 - ? \end{pmatrix}$
Fill in Z using the following $\frac{n}{2} \times \frac{n}{2}$ matrices:
$P_1 = A(H - O')$,
$P_2 = (A + L)O'$,
$P_3 = (G + O)T$,
$P_4 = O(E - T)$,
$P_5 = (A + O)(T + O')$
$P_6 = (L - O)(E + O')$,
$P_7 = (A - G)(T + H)$

then show that time complexity for finding Z can get significantly better than in part (b).
(3 Points)

Exercise 2: K-th Smallest Element *(5 Points)*

Consider two arrays A and B of unique integers such that they are sorted in an increasing order and of size m and n, respectively. Let $k \leq m, n$, give an algorithm that finds the k-th smallest integer in the sorted union of the two arrays in time $O(\log k)$. Argue correctness and analyze its running time.
Remark: for simplicity, one may consider k to be a power of two.

Exercise 3: Triangle with Shortest Perimeter *(7 Points)*

Let $P = \{(x_i, y_i) \in \mathbb{R}^2 \mid i = 1, \dots, n\}$ be a set of points in $\mathbb{R}^2$. Given three distinct points $a, b, c \in P$ they span a triangle with *perimeter*

$$peri(a, b, c) = d(a, b) + d(b, c) + d(a, c),$$

where $d(\cdot, \cdot)$ determines the euclidean distance of two points.

(a) Assume e to be the shortest perimeter of a triangle formed by 3 points from P, how many triangles (formed by 3 points from P) can there be in a rectangle of size $e/5 \times e/4$? *(2 Points)*

(b) Describe a $\mathcal{O}(n \cdot \log(n))$ algorithm which finds the smallest triangle perimeter in P. Argue shortly the correctness of your algorithm and its running time. *(5 Points)*